# Change of Canonical Structure in Chern-Simons QED$_3$ by Quantum Effect

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**Abstract**

In a three-dimensional Abelian gauge theory with the Chern-Simons term as a kinetic term, we investigate how the canonical structure is affected by a quantum effect. An equal-time commutator for gauge field is evaluated using the Bjorken-Johnson-Low limit. We find that the commutator is modified in a non-trivial way and that there appears a total sign ambiguity in a newly derived term due to limiting processes of the Bjorken-Johnson-Low formula, which is specific in odd dimensional space-time.

**Key Words : Canonical structure, Chern-Simons QED$_3$, Bjorken-Johnson-Low limit, Parity anomaly**

## 1. Introduction

The canonical structure plays a very important role in quantum theories. The structure is given by a starting Lagrangian and is not changed usually. It decides a nature of particles under consideration as statistics for example. But in some specific cases, there appears abnormal changing of the canonical structure by dynamics.

In a context of investigating quantum field theories, non-canonical terms sometimes appear in equal-time commutators. Well-known example is the Schwinger terms in fermion current-current equal-time commutation relations. Quantum corrections induce additional terms which cannot be absorbed by local counter terms. Thus quantum effects change the equal-time commutators in these theories. [1]

There are several discussions to explain why those anomalous terms may appear, using mathematical concept, cocycle [2], or Berry's phase and so on [3, 4, 5]: The anomalous or non-canonical term has a topological origin, i.e., topological non-triviality in gauge orbit space or determinant line bundle. The non-triviality induces "anomalies", "Schwinger terms" or "deformation of simplectic structures". Here one may have natural question: Is it only the case?

In this paper, we present a novel "anomalous" commutator, which seems to have an entirely different origin from anomalous commutators known previously. We treat three-dimensional Abelian gauge theory with the Chern-Simons term as kinetic term of gauge field and without usual quadratic derivative term, coupled to a two component massless fermion [6]. This is the quantum electrodynamics (QED) only with Chern-Simons term as the Lagrangian of the gauge field in three-dimensional space-time. We call this theory "Chern-Simons QED$_3$".

## 2. Chern-Simons QED$_3$

The Lagrangian of the Chern-Simons QED$_3$ is given by

$$\mathcal{L} = \frac{\theta}{2}\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \bar{\psi}\gamma^\mu(i\partial - eA)_\mu\psi \ , \tag{1}$$

where the repetition of indices means to take the summation over $0,1,2$ . The $\varepsilon^{\mu\nu\rho}$ is a totally anti-symmetric tenser with $\varepsilon^{012} = 1$ . The Dirac's $\gamma$-matrices are defined by $\gamma^o = \sigma^3, \gamma^1 = i\sigma^1, \gamma^2 = i\sigma^2$ by using the Pauli matrices $\sigma^1, \sigma^2$ and $\sigma^3$. We use the Minkowski metric so as $\mathrm{diagonal}(g^{\mu\nu}) = (1, -1, -1)$. Mass dimensions of $\theta$ and $e$ are $[\theta] = [mass]^1$ and $[e] = [mass]^{1/2}$.

If we try to quantize the Chern-Simons QED$_3$ in the Hamiltonian formalism, there appear many constraints. The theory is a constraint system. In that case, we may quantize the theory following two strategies: (i) One method is to construct a generalized Hamilton system following the Dirac's program [7]. After that, we pass to quantum theory regarding the

Dirac brackets as equal-time (anti-) commutator. (ii) On the other hand, according to Faddeev-Senjanovic method for a constraint system [8], we can quantize the theory in path-integral formalism and obtain Feynman rules. Now we would like to know the canonical structure of the theory. There is a traditional method to derive equal-time commutation relations from given Feynman rules. This is the Bjorken-Johnson-Low limit. [5] We can derive equal-time commutators through the Bjorken-Johnson-Low limit.

Usually, both prescriptions of quantization for a constraint system mentioned above is coincide with each other. By finite renormalizations, the radiative correction is absorbed in coupling constants, wave functions, and mass parameters multiplicatively. In the exceptional cases where theories have anomalies, there appear the Schwinger terms.

How about Chern-Simons $QED_3$? We apply the Bjorken-Johnson-Low method to Chern-Simons $QED_3$ and show that an equal-time commutation relation is modified from the one derived in Dirac's formalism, in a manner that an additional term cannot be absorbed by the usual renormalization. This is also the case that the canonical structure is changed due to quantum effect. But it is important to note that the "anomalous" term in Chern-Simons $QED_3$ seems to have a different origin from the usual Schwinger terms.

Let us consider to quantize Chern-Simons $QED_3$. Eq. (1) is gauge invariant (apart from a total derivative term) so that we should add the covariant gauge fixing term

$$\mathcal{L} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \quad , \tag{2}$$

where $\xi$ is a gauge fixing parameter. Even if we add the covariant gauge fixing term to eq. (1), constraints cannot be removed completely so that we use the Dirac's procedure for quantization of constraint system. After routine works we obtain Dirac brackets. Especially we are interested in the following bracket.

$$\{A^i(t,\vec{x}),\ A^j(t,\vec{y})\}_{Dirac} = \frac{1}{\theta}\varepsilon^{ij}\delta(\vec{x},-\vec{y}) \ , \tag{3}$$

where $i,j = 1,2$. The usual procedure of quantization says that we should replace eq. (3) with the equal-time commutation relation,

$$[A^i(t,\vec{x}), A^j(t,\vec{y})] = \frac{i}{\theta}\varepsilon^{ij}\delta(\vec{x}-\vec{y}) \ . \tag{4}$$

It should be noted that the right hand side of eq. (4) is anti-symmetric for exchange of i and j.

## 3. Bjorken-Johnson-Low limit

The problem is whether any quantum effects change the equal-time commutator or not. We examine this by using the Bjorken-Johnson-Low limit, in which the equal-time commutator is given by the formula (Appendix A),

$$i\int d\vec{x}\, e^{-i\vec{q}\cdot(\vec{x}-\vec{y})} < \alpha|\left[A^i(t,\vec{x}), A^j(t,\vec{y})\right]|\beta >$$

$$= \lim_{q_0\to\infty} q_0 D^{ij}(q) \ , \tag{5}$$

where

$$D^{ij}(q) \equiv \int d^3x\, e^{i\,q\cdot(x-y)} < \alpha|TA^i(x)\, A^j(y)|\beta > \ . \tag{6}$$

It is assumed that $q_0 D^{ij}(q)$ in eq. (5) is an analytic function. Eq. (6) is photon propagator in energy-momentum space. By using the Bjorken-Johnson-Low limit, we can derive equal-time commutators from Feynman rule. In a sense, the Bjorken-Johnson-Low formula, eqs. (5) and (6), is the definition of equal-time commutators.

Consider free case. The free photon propagator is given by

$$D^{ij}_{free}(q) = -\frac{1}{\theta}\varepsilon^{ij}\frac{q^0}{q^2} - i\xi\frac{q^i q^j}{(q^2)^2} \quad , \tag{7}$$

where $i,j = 1,2$. Substituting eq. (7) into eq. (5) we recover eq. (4) as expected. Thus in tree level the commutator defined by using the Bjorken-Johnson-Low limit is consistent with the one obtained in Dirac procedure.

## 4. Fermion Loop Correction

Under the parity transformation [9], the gauge part in eq. (1) is odd (i.e., changes its total sign) and the fermion part is even. Further if we consider a radiative correction due to fermion loop, both parity-even and odd parts are induced in an effective action of gauge field. The appearance of parity-odd part, which is the induced Chern-Simons term, is called “parity anomaly” [10] and discussed extensively in other contexts. Rather, our attention in this paper is concentrated to the induced parity-even part. Seeing from the side of gauge field, while there is no parity-even part in the starting Lagrangian, the quantum correction induces parity-even part through fermion loop correction. This is just the essence of our novel “anomalous" term.

Now we include fermion loop correction as

$$D^{ij}(q) = [D^{ij}_{free}(q)^{-1} - \Pi^{ij}(q)]^{-1} \tag{8}$$

where $\Pi^{ij}(q)$ is vacuum polarization tensor of gauge field. Up to one-loop, an explicit calculation shows

$$\Pi^{ij}(q) = \frac{e^2}{16}\left(q^2 g^{ij} - q^i q^j\right)\frac{1}{|q|} + \frac{e^2}{4\pi}\varepsilon^{ij} q_0 \tag{9}$$

The first term in the right hand side of eq. (9) is parity-even and the second is odd. The odd part is induced by introducing a heavy fermion as a regulator of ultra-violet divergence. This term is the parity anomaly and has a topological origin. (We have used the Pauli-Villars regularization. There is a regularization ambiguity [9, 10] in the last term of eq. (9), which, however, does not change our main results.) Then the propagator of gauge field including effects of fermion loop becomes

$$D^{ij}(q) = -\frac{\left(\frac{e^2}{16}\right)^2}{\left(\theta+\frac{e^2}{4\pi}\right)^2+\left(\frac{e^2}{16}\right)^2}\left(g^{ij} - \frac{q^i q^j}{q^2}\right)\frac{1}{|q|}$$

$$-\frac{\theta+\frac{e^2}{4\pi}}{(\theta+\frac{e^2}{4\pi})^2+(\frac{e^2}{16})^2}\varepsilon^{ij}\frac{q^0}{q^2} - i\xi\frac{q^i q^j}{(q^2)^2}\ , \tag{10}$$

by using eqs. (8) and (9). (Appendix B) In the limit as $q^0 \gg |\vec{q}|$, we have

$$D^{ij}(q) \xrightarrow{q^0 \gg |\vec{q}|} -\frac{\left(\frac{e^2}{16}\right)^2}{\left(\theta+\frac{e^2}{4\pi}\right)^2+\left(\frac{e^2}{16}\right)^2} g^{ij}\frac{1}{q^0}$$

$$-\frac{\theta+\frac{e^2}{4\pi}}{(\theta+\frac{e^2}{4\pi})^2+(\frac{e^2}{16})^2}\varepsilon^{ij}\frac{1}{q^0} + O((\tfrac{1}{q^0})^2). \tag{11}$$

Here we substitute eq. (11) into eq. (5) and find that the equal-time commutation relation derived through Bjorken-Johnson-Low limit is

$$\left[A^i(t,\vec{x}), A^j(t,\vec{y})\right] = i\frac{\theta+\frac{e^2}{4\pi}}{(\theta+\frac{e^2}{4\pi})^2+(\frac{e^2}{16})^2}\varepsilon^{ij}\delta(\vec{x}-\vec{y})$$

$$+i\frac{\left(\frac{e^2}{16}\right)^2}{(\theta+\frac{e^2}{4\pi})^2+(\frac{e^2}{16})^2} g^{ij}\delta(\vec{x}-\vec{y})\ . \tag{12}$$

Of course, if we set the coupling constant $e$ to zero, eq. (12) reduces to the free case eq. (4).

In eqs. (5), (6), and (10), we obtain the relation $\frac{q^0}{|q|} = \frac{q^0}{|q^0|} = \mathrm{sgn}(q^0)$ for $q^0 \gg |\vec{q}|$, where sgn means the signum function. In the limit $q_0 \to \infty$, $q^0$ in eq. (11) is positive so that $\frac{q^0}{|q^0|} \to 1$. This behavior in the limiting process gives us the result in eq. (12).

## 5. Change of Canonical Structure

It is a novel feature that there appears the symmetric part proportional to $g^{ij}$ as a correction by the quantum effect, while the free case is consist of only totally anti-symmetric part proportional to $\varepsilon^{ij}$. The symmetric part cannot be absorbed by a finite renormalization of coupling constant or field operator, because the tensor structure is altered. If we tend to renormalize multiplicatively, the renormalization factor $Z$ acquires to have the tensor structure like $Z_{ij}$, which is unusual. Thus a quantum effect induces a kind of "anomalous" term. Further, naively seeing, the "anomalous" term does not seem to have topological origin as the parity anomaly f QED$_3$ [11].

We may consider more higher radiative corrections. In the case of the usual QED$_3$, the non-renormalization theorem holds for the part of the parity anomaly [11, 12]. We can extend the theorem to the case of Chern-Simons QED$_3$. On the other hand, our "anomalous" term has its origin in the parity even part so that there is not such a theorem. Therefore more higher-order loops may induce more corrections for the commutator. Eq. (12) is obtained starting from the massless fermion. In the case of massive fermion, we have the same expression as eq. (12).

## 6. Discussions and Conclusions

The result of eq. (12) has a curious aspect. If we interpret the left hand side as the usual commutator of $A^i$'s, the side is totally anti-symmetric under the exchange of (i, x) and (j, y), but the right hand side is not totally anti-symmetric because of the term proportional to $g^{ij}$. This is seen typically, if we set i = j. Then eq. (12) becomes

$$\left[A^i(t,\vec{x}), A^i(t,\vec{y})\right] = i\frac{(\frac{e^2}{16})^2}{(\theta+\frac{e^2}{4\pi})^2+(\frac{e^2}{16})^2} g^{ii}\delta(\vec{x}-\vec{y}) \tag{13}$$

Naively thinking, the equal-time commutator of the same field operators should vanishes. But precisely speaking, we should denote the left hand side as a vacuum expectation value

$$< 0\left|\left[A^i(t,\vec{x}), A^i(t,\vec{y})\right]\right|0 > \tag{14}$$

so that eq. (12) can become understandable in this sense. There is no reason that the expectation value of the commutator between the same variables should vanish.

The classical algebra for canonical variables which for quantum operators. In the quantum level, there might image a new algebra. Then the result seems to suggest a following important property of Chern-Simons QED$_3$: The change of algebra for canonical variables can be interpreted as the change of statistics. [13, 14] Thus quantum effects change the statistics of the quanta. Such a situation has known in the past. Some kinds of bound states can change those statistics from the one of each component field. But previously we do not know the case like ours, where the most basic commutation relation for canonical variables is changed in a nontrivial way as denoted typically in eq. (13).

As a phenomenological model of the high-T$_c$ superconductivity, three dimensional O(3) non-linear sigma model with the Hopf term [13] was discussed. In low energy regions, the model becomes CP$^1$ model with the Chern-Simons term for a hidden U(1) gauge field ( Chern-Simons CP$^1$) [15, 16]. It is also a theory with the Chern-Simons term as kinetic term. While there is the difference that the Chern-Simons QED$_3$ is coupled to fermion but the Chern-Simons CP$^1$ is coupled to complex scalar field, the situation is similar. There may occur the changing of canonical structure due to quantum effect in Chern-Simons CP$^1$. Then the Bose-Fermi transmutation shown in the Chern-Simons CP$^1$, might be affected by the change of the canonical structure.

In eq. (11), we have used the traditional manipulations which are employed in the usual Bjorken-Johnson-Low limit calculus. Here it should be noted that there is an ambiguity by the limiting process of $q_0$ , which is a new aspect of the Bjorken-Johnson-Low limit in odd dimensional space-time. Usually the terms obtained after taking limits $q_0 \to \infty$ and $q_0 \to -\infty$ coincide with each other. However the limits give us the term with opposite sign in our case. Thus there is the ambiguity of the total sign under the limiting processes as $q_0 \to \pm\infty$. This is the unusual behavior which is never seen in even dimensional space-time. The essence is the analytic behavior of vacuum polarization tensor in odd dimensions. The function like $\frac{1}{|q|}$ is allowed to appear because the coupling constant $e$ has the mass dimension $[mass]^{1/2}$. From a physical point of view, this ambiguity is not serious. We can decide the sign so as an experimental setting can be realized by the phenomenological model.

In conclusion, we have shown that the non-canonical term in the equal-time commutator between gauge fields appears in the Chern-Simons QED$_3$ which has been derived by using the Bjorken-Johnson-Low limit. The implication of the result is that the statistics may be affected by the quantum effect. Further the Bjorken-Johnson-Low limit in odd dimensions has the ambiguity in its limiting process, which may make the structure of the theory richer.

## Appendix A: The Bjorken-Johnson-Low formula

We consider two field operators $A(t,\vec{x})$ and $B(t,\vec{y})$. A matrix element of the T-product of these operators is defined by

$$D(q) \equiv \int d^3x\, e^{iq\cdot(x-y)} < \alpha|\, TA(x)B(y)|\beta > \quad , \qquad \text{(A.1)}$$

where $< \alpha|$ and $|\beta >$ are quantum state vectors. The expectation vale depends on $x-y$ because of its translational invariance so that the y-dependence in the left hand side of eq. (A.1) vanishes by the x-integration. Under the condition $q_0 \neq 0$, eq. (A.1) is rewritten as

$$D(q) \equiv -i\frac{1}{q_0}\int d^3x \left(\frac{\partial}{\partial x_0} e^{iq\cdot(x-y)}\right) < \alpha|TA(x)B(y)|\beta > \ . \qquad \text{(A.2)}$$

The time integration by parts may produce the surface term which can be dropped because the field operators vanish at the infinite surface. Then we obtain

$$q_0 D(q) = i\int d^3x\, e^{iq\cdot(x-y)} < \alpha|T\left(\frac{\partial}{\partial x_0}A(x)\right)B(y)|\beta >$$
$$+\, i\int d\vec{x}\, e^{-i\vec{q}\cdot(\vec{x}-\vec{y})} < \alpha|[A(t,\vec{x}),B(t,\vec{y})|\beta > \ . \qquad \text{(A.3)}$$

The second term in the right hand side in eq. (A.3) appears because of the T-product. The first term should vanish in $q_0 \to \infty$ limit according to the Riemann-Lesbegue lemma. Thus eq. (A.3) reduces to

$$\lim_{q_0\to\infty} q_0\, D(q) =$$
$$i\int d\vec{x}\, e^{-i\vec{q}\cdot(\vec{x}-\vec{y})} < \alpha|\,[A(t,\vec{x}),B(t,\vec{y})]|\beta > \quad . \qquad \text{(A.4)}$$

This is the general form of the Bjorken-Johnson-Low formula.

## Appendix B: The propagator of gauge field including the vacuum polarization effect

We show the propagator of the gauge field including the vacuum polarization effect, whose Lagrangian is given only by Chern-Simons term. The free propagator is

$$D_{free}^{\mu\nu}(q) = -\frac{1}{\theta}\varepsilon^{\mu\nu\rho}\frac{q_\rho}{q^2} - i\xi\frac{q^\mu q^\nu}{(q^2)^2}\ , \qquad \text{(B.1)}$$

and its inverse becomes

$$D_{free}^{-1\ \mu\nu}(q) = -\theta\varepsilon^{\mu\nu\rho}q_\rho - i\xi q^\mu q^\nu\ , \qquad \text{(B.2)}$$

in the energy-momentum space. The vacuum polarization tensor is calculated as

$$\Pi^{\mu\nu}(q) = \frac{e^2}{16}(q^2 g^{\mu\nu} - q^\mu q^\nu)\frac{1}{|q|} + \frac{e^2}{4\pi}\varepsilon^{\mu\nu\rho}q_\rho\ , \qquad \text{(B.3)}$$

up to one-loop perturbation. The propagator including the effect of the vacuum polarization is obtained by

$$D^{\mu\nu} = \left[D_{free}^{-1\ \mu\nu} - \Pi^{\mu\nu}\right]^{-1}\ . \qquad \text{(B.4)}$$

Eventually we have the result as

$$D^{\mu\nu}(q) = -\frac{\left(\frac{e^2}{16}\right)^2}{\left(\theta+\frac{e^2}{4\pi}\right)^2+(\frac{e^2}{16})^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\frac{1}{|q|}$$

$$-\frac{\theta+\frac{e^2}{4\pi}}{(\theta+\frac{e^2}{4\pi})^2+(\frac{e^2}{16})^2}\varepsilon^{\mu\nu\rho}\frac{q_\rho}{q^2} - i\xi\frac{q^i q^j}{(q^2)^2}\ . \qquad \text{(B.5)}$$

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【日本語要旨】

# Change of Canonical Structure in Chern-Simons $QED_3$ by Quantum Effects

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アーベル群上のチャーン・シモンズ項を力学的な作用に持つ3次元量子力学において、正順構造が量子効果によってどのような影響を受けるかを研究する。ゲージ場の同時刻交換関係をブジョルケン、ジョンソン、ロー極限の手法で評価する。奇数時限時空の特性として、同時刻交換関係が非自明に修正され正準構造が修正されることを導いた。そのとき、新しく導かれた項にはブジョルケン、ジョンソン、ロー公式における極限操作に起因した全体符号についての任意性が現れることを示した。