Quantized Pure U(1) Gauge Field Theory on Finite Element in (d+1)-Dimensional Space-Time

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Abstract

We construct a quantized pure U(1) gauge field theory on a finite element in (d+1)-dimensional space-time. The field equations of motion are formulated on the finite element and it is shown that the equations have a desirable dispersion relation. We show that the field equations on the finite element preserve the symmetry which corresponds to the gauge symmetry of the continuum theory and that a gauge fixing is needed. Further, we solve the field equations of motion and prove that the consistency condition for the quantization is satisfied using the solution.

Key Words : Gauge field theory, Finite element, Canonical quantization, Consistency

1. Introduction

At the present the lattice gauge theory [1] plays an important role in understanding the non-perturbative properties of fundamental interactions. However there exists the so-called doubling problem for fermions. That is, when the fermion field is formulated on a lattice, there appears the excess of the species. To resolve this problem, many efforts have been made. [2][3][4]

Bender, et al. [5][6] applied the method of finite elements to (1+1)-dimensional scalar and spinor fields. For the (1+1)-dimensional spinor field, it was shown that there was no doubling and the consistency condition on the quantization was satisfied.

We extend their ideas to the case in the (d+1)-dimensional space-time. [7][8] That is, we showed that the Dirac equation on the finite element in the (d+1)-dimensional space-time was free from the doubling problem. Further, we found the solution of the equation and proved directly that the consistency condition on the quantization was satisfied.

After these studies, the method has been applied to a tunneling matrix method, an operator ordering Problem. [9]

On the other hand, it may be interesting to investigate whether the method of finite element can be applied to the gauge field. The major motivations of its application are following:

(1) The method of finite element is considered as a sort of the systematic procedure to construct the regularized version of a continuum. One of the merits in the procedure is that the spinor field is formulated without doubling. From the unified point of view, we should formulate the gauge field also by using the method of finite element.

(2) The standard lattice gauge theory has a compact gauge field space whi1e the continuum gauge theory has a non-compact one. There might be a possibility that both theories have an essential difference. If we can construct the 1attice gauge theory with non-compact gauge fie1d space, the theory might be more suitab1e as the regu1arized version for the corresponding continuum theory.

In this paper, we construct the non-compact pure U(l) gauge fie1d on the finite e1ement in the (d+1)-dimensional space-time. In Sec. 2, we explain the method of finite element and show how to apply it to the quantum field theory. In Sec. 3, we summarize the continuum pure U(1) gauge theory. The equation of motion on the finite e1ement is set up in Sec. 4. In Sec. 5, we derive the dispersion re1ation. In Sec. 6, the gauge symmetry on the finite element and the necessity of the gauge fixing term are discussed. We solve the equation of motion in Sec. 7. In Sec. 8, the equal- time commutation relations on the finite element and the explicit statement of the consistency condition are shown. In Sec. 9, we prove the consistency condition to hold. Finally, we conclude in Sec. 10.

The novel point of our formalism is as follows. The theory in the continuous space-time is described by the gauge field which has its value on the group algebraic space. We call this field as being non-compact. But in the lattice space-time the gauge field is introduced as the group valued link variable. It is because the previously known difference operator breaks the gauge invariance if we use the algebraic formalism. Therefore there may appear any changing of properties for the theory in the discrete space-time which does not exist in the continuous space-time. In this paper, we propose a new formalism on discrete space-time in which the gauge field has the gauge invariance even if its value is on algebraic space. This is the most new aspect of this work. In addition, we also give a proof that the formalism is consistent in the quantization. Thus we propose the algebraic valued gauge field, which is gauge invariant and also is consistent with the quantization.

2. The method of finite element

Originally, the method of finite element is used in order to solve partial differential equations in computer calculus. When we solve the coupled partial differential equations with boundary conditions, the method gives us a powerful prescription. The method consists of the following four steps :

Step 1: Divide the domain where the partial differential equations are set up, into patches called "finite element".

Step 2: Approximate the solution of the partial differential equations by lower order polynomials.

Step 3: Require that the partial differential equations hold at any one point on each finite element.

Step 4: Solve the algebraic equations for the coefficients of the polynomials.

In applying the method to the quantum field theory, we simply replace (a) "domain" with "space-time", (b) "partial differential equations" with "field equations", and (c) "algebraic equations for the coefficients of the polynomials" with "time evolution equations of field operators". Further the method should be consistent with the quantum condition on the field operators. Thus the formulation that will be constructed should satisfies the following consistency condition: The equal time (anti-) commutation relations are invariant under the time evolution.

The procedures mentioned above are a most general scenario. In this paper, we restrict ourselves to adopt a (d+1)-dimensional parallelepiped as a patch and make a linear approximation to a field operator. And we require that field equations hold at the center of each finite element.

The finite elements are introduced as follows:
A $\{n_0,\dots,n_i,\dots,n_d\}$-element is the parallelepiped that has the domain $\{(x_0,\dots,x_i,\dots,x_d)|x_i=(n_i-1)a_i+\alpha_i, 0\leqq\alpha_i\leqq a_i,\ i=0,1,\dots,d\}$ where a_i's are spacing for each space-time direction and α_i's are continuous parameters running from 0 to a_i.

The linear approximation for the field operators $f(x)$ is written as

$$f(x)=\sum_{\{\varepsilon_0,\varepsilon_1,\dots,\varepsilon_d\}}\prod_{s=0}^{d}\left\{\varepsilon_s+(-1)^{\varepsilon_s}\frac{\alpha_s}{a_s}\right\} \times f(n_0-\varepsilon_0,n_1-\varepsilon_1,\dots,n_d-\varepsilon_d) \qquad (2.1)$$

where ε_s's run 0 and 1 for each $s=0,1,\dots,d$. For the simplicity of notation, we denote $\{\varepsilon_0,\varepsilon_1,\dots,\varepsilon_d\}$ as $\{\varepsilon\}$ and $f(n_0-\varepsilon_0,n_1-\varepsilon_1,\dots,n_d-\varepsilon_d)$ as $f(n-\varepsilon)$ henceforth. Thus eq. (2.1) is denoted by

$$f(x)=\sum_{\{\varepsilon\}}\prod_{s=0}^{d}\left\{\varepsilon_s+(-1)^{\varepsilon_s}\frac{\alpha_s}{a_s}\right\}f(n-\varepsilon)\quad . \qquad (2.2)$$

3. The equation of motion and equal time commutation relations in the continuum theory

We summarize the equation of motion and equal time commutation relations of the U(1) gauge field in the (d+1)-dimensional continuous space-time for later use.

The Lagrangian density is

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2\xi}(\partial_\mu A^\mu)^2\ , \qquad (3.1)$$

where $F_{\mu\nu}==\partial_\mu A_\nu-\partial_\nu A_\mu$, $(\mu,\nu=0,1,\dots,d)$ and the repetition of index means the sum from 0 to d. We use Minkowski metric $g^{\mu\mu}=(+1.-1,\dots,-1)$ and $g^{\mu\nu}=0$ for $\mu\neq\nu$. The second term in eq. (3.1) is a gauge fixing term and ξ is a covariant gauge fixing parameter. In the continuum theory, this term is required to fix the gauge. In Sec. 5, we will discuss the gauge symmetry of the U(1) gauge field on finite elements.

The equation of motion is the "second" order partial differential equation,

$$\partial_\nu \partial^\nu A^\mu - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial_\nu A^\nu = 0 \ . \qquad (3.2)$$

As mentioned in Sec. 2, we perform the linear approximation for field operators. In order that the linear approximation is meaningful, we rewrite eq. (3.2) to a set of the coupled, "first" order partial differential equations,

$$E_\nu^\mu = \partial_\nu A^\mu, \qquad (3.3)$$

$$\partial^\nu E_\nu^\mu - \left(1 - \frac{1}{\xi}\right) \partial^\mu E_\nu^\nu = 0 \ . \qquad (3.4)$$

The canonical quantization method is applied to this system. The conjugate momentum of A^ν is, from eq. (3.1),

$$\pi^\mu = \partial^\mu A^0 - \partial^0 A^\mu - g^{\mu 0} \frac{1}{\xi} \partial_\nu A^\nu \qquad (3.5)$$

For the dynamical variables A^μ and π^μ, the canonical equal time commutation relations are

$$[A^\mu(t,\vec{x}), \pi^\nu(t,\vec{y})] = i g^{\mu\nu} \delta^{(d)}(\vec{x} - \vec{y}) \ , \qquad (3.6)$$

$$[A^\mu(t,\vec{x}), A^\nu(t,\vec{y})] = [\pi^\mu(t,\vec{x}), \pi^\nu(t,\vec{y})] = 0 \ , \qquad (3.7)$$

where $\vec{x}$, $\vec{y}$ are d-dimensional vectors and $\delta^{(d)}(\vec{x} - \vec{y})$ is the d-dimensional Dirac δ-function. A set of eqs. (3.3)-(3.7) describes the quantized U(1) gauge field theory.

In eq. (3.3), we have introduced E_ν^μ. The equal time commutation relations between E_ν^μ and A^μ, E_ν^μ and π^μ, E_ν^μ itself, are derived from eqs. (3.6) and (3.7) using eqs. (3.3) and (3.5). The results are

$$[A^\mu(t,\vec{x}),\, E_0^0(t,\vec{y})] = -i\xi g^{\mu 0} \delta^{(d)}(\vec{x} - \vec{y}) \ , \qquad (3.8a)$$

$$\left[A^\mu(t,\vec{x}),\, E_0^i(t,\vec{y})\right] = -i g^{\mu i} \delta^{(d)}(\vec{x} - \vec{y}) \ , \qquad (3.8b)$$

$$[A^\mu(t,\vec{x}),\, E_i^\nu(t,\vec{y})] = 0 \ , \qquad (3.8c)$$

for E_ν^μ and A^μ,

$$[E_0^0(t,\vec{x}),\, \pi^0(t,\vec{y})] = 0 \ , \qquad (3.9a)$$

$$\left[E_0^0(t,\vec{x}),\, \pi^i(t,\vec{y})\right] = -i \partial_i^y \delta^{(d)}(\vec{x} - \vec{y}) \ , \qquad (3.9b)$$

$$\left[E_0^i(t,\vec{x}),\, \pi^\nu(t,\vec{y})\right] = i g^{0\nu} \partial_i^y \delta^{(d)}(\vec{x} - \vec{y}) \ , \qquad (3.9c)$$

$$\left[E_i^\rho(t,\vec{x}),\, \pi^\nu(t,\vec{y})\right] = -i g^{\rho\nu} \partial_i^y \delta^{(d)}(\vec{x} - \vec{y}) \ , \qquad (3.9d)$$

for E_ν^μ and π^μ, and

$$[E_0^0(t,\vec{x}),\, E_0^0(t,\vec{y})] = 0 \ , \qquad (3.10a)$$

$$\left[E_0^0(t,\vec{x}),\, E_0^i(t,\vec{y})\right] = -i(\xi - 1) \partial_i^y \delta^{(d)}(\vec{x} - \vec{y}) \ , \qquad (3.10b)$$

$$\left[E_0^i(t,\vec{x}),\, E_0^j(t,\vec{y})\right] = 0 \ , \qquad (3.10c)$$

$$\left[E_i^\rho(t,\vec{x}),\, E_0^0(t,\vec{y})\right] = i\xi g^{\rho 0} \partial_i^y \delta^{(d)}(\vec{x} - \vec{y}) \ , \qquad (3.10d)$$

$$\left[E_j^\rho(t,\vec{x}),\, E_0^i(t,\vec{y})\right] = i g^{\rho i} \partial_j^y \delta^{(d)}(\vec{x} - \vec{y}) \ , \qquad (3.10e)$$

$$\left[E_i^\rho(t,\vec{x}),\, E_j^\nu(t,\vec{y})\right] = 0 \ , \qquad (3.10f)$$

for E_ν^μ itself.

4. The equation of motion on the finite element

According to the prescription mentioned in Sec. 2, we formulate the (d+1)-dimensional U(1) gauge field on the finite element by using eq. (2.2).

The linear approximations for $A^\mu(x)$ and $E_\nu^\mu(x)$ are

$$A^\mu(x) = \sum_{\{\varepsilon\}} \prod_{s=0}^d \left\{\varepsilon_s + (-1)^{\varepsilon_s} \frac{\alpha_s}{a_s}\right\} A^\mu(n - \varepsilon) \ , \qquad (4.1)$$

$$E_\nu^\mu(x) = \sum_{\{\varepsilon\}} \prod_{s=0}^d \left\{\varepsilon_s + (-1)^{\varepsilon_s} \frac{\alpha_s}{a_s}\right\} E_\nu^\mu(n - \varepsilon) \ . \qquad (4.2)$$

We require eqs. (3.3) and (3.4) to hold at the center of the finite elemente $\alpha_s = \frac{a_s}{2}$, $(s = 0, 1, \cdots, d)$. We substitute eqs. (4.1) and (4.2) into eqs. (3.3) and (3.4), and obtain the coupled equations for $A^\mu(x)$ and $E_\nu^\mu(x)$,

$$\sum_{\{\varepsilon\}} \frac{1}{2} E_0^\mu(n - \varepsilon) = \sum_{\{\varepsilon\}} \frac{(-1)^{\varepsilon_0}}{a_0} A^\mu(n - \varepsilon) \ , \qquad (4.3)$$

$$\sum_{\{\varepsilon\}} \frac{1}{2} E_i^\mu(n - \varepsilon) = \sum_{\{\varepsilon\}} \frac{(-1)^{\varepsilon_i}}{a_i} A^\mu(n - \varepsilon) \ , \qquad (4.4)$$

$$\frac{1}{\xi} \sum_{\{\varepsilon\}} \frac{(-1)^{\varepsilon_0}}{a_0} E_0^0(n - \varepsilon) - \sum_{j=1}^d \sum_{\{\varepsilon\}} \frac{(-1)^{\varepsilon_j}}{a_j} E_j^0(n - \varepsilon)$$

$$- \left(1 - \frac{1}{\xi}\right) \sum_{\{\varepsilon\}} \frac{(-1)^{\varepsilon_0}}{a_0} \sum_{j=0}^d E_j^j(n - \varepsilon) = 0 \ , \qquad (4.5)$$

$$\sum_{\{\varepsilon\}} \frac{(-1)^{\varepsilon_0}}{a_0} E_0^i(n - \varepsilon) - \sum_{j=0}^d \sum_{\{\varepsilon\}} \frac{(-1)^{\varepsilon_j}}{a_j} E_j^i(n - \varepsilon)$$

$$+ \left(1 - \frac{1}{\xi}\right) \sum_{\{\varepsilon\}} \frac{(-1)^{\varepsilon_i}}{a_i} \{E_0^0(n - \varepsilon) + \sum_{j=1}^d E_j^j(n - \varepsilon)\} = 0 \ . \qquad (4.6)$$

In deriving the above equations, we put $\alpha_s = \frac{a_s}{2}$ and used an identity, $\left\{\varepsilon_s + (-1)^{\varepsilon_s} \frac{1}{2}\right\} = \frac{1}{2}$, which holds irrespective of ε_s. From eqs. (4.3)-(4.6), we derive two types of the coupled equations. The first one shall be used for obtaining the one-time-step solution, and the other shall be used for discussing the gauge symmetry and the dispersion relation.

4.1. Time evolution equations for A^μ and E_μ^ν

We separate the terms at time n_0 and the ones at time $n_0 - 1$ in eqs. (4.3)-(4.6). Then we obtain the following time evaluation equations of A^μ and E_ν^μ,

$$\sum_{\vec{\varepsilon}} \{ \frac{1}{2} E_0^\mu(n_0, \vec{n} - \vec{\varepsilon}) - \frac{1}{a_0} A^\mu(n_0, \vec{n} - \vec{\varepsilon})\}$$

$$= \sum_{\vec{\varepsilon}} \{ -\frac{1}{2} E_0^\mu(n_0 - 1, \vec{n} - \vec{\varepsilon}) - \frac{1}{a_0} A^\mu(n_0 - 1, \vec{n} - \vec{\varepsilon})\}, \qquad (4.7)$$

$$\sum_{\vec{\varepsilon}} \{ \frac{1}{2} E_i^\mu(n_0, \vec{n} - \vec{\varepsilon}) - \frac{(-1)^{\varepsilon_i}}{a_i} A^\mu(n_0, \vec{n} - \vec{\varepsilon})\}$$

$$= \sum_{\vec{\varepsilon}} \{ -\frac{1}{2} E_i^\mu(n_0 - 1, \vec{n} - \vec{\varepsilon}) + \frac{(-1)^{\varepsilon_i}}{a_i} A^\mu(n_0 - 1, \vec{n} - \vec{\varepsilon})\}, \qquad (4.8)$$

$$\sum_{\vec{\varepsilon}}\{\frac{1}{\xi}\frac{1}{a_0}E_0^0(n_0,\vec{n}-\vec{\varepsilon})-\sum_{j=1}^{d}\frac{(-1)^{\varepsilon_j}}{a_j}E_j^0(n_0,\vec{n}-\vec{\varepsilon})$$

$$-(1-\frac{1}{\xi})\frac{1}{a_0}\sum_{j=1}^{d}E_j^j(n_0,\vec{n}-\vec{\varepsilon})\}\ ,$$

$$=\sum_{\vec{\varepsilon}}\{\frac{1}{\xi}\frac{1}{a_0}E_0^0(n_0-1,\vec{n}-\vec{\varepsilon})+\sum_{j=1}^{d}\frac{(-1)^{\varepsilon_j}}{a_j}E_j^0(n_0-1,\vec{n}-\vec{\varepsilon})$$

$$-(1-\frac{1}{\xi})\frac{1}{a_0}\sum_{j=1}^{d}E_j^j(n_0-1,\vec{n}-\vec{\varepsilon})\},\qquad(4.9)$$

$$\sum_{\vec{\varepsilon}}\{\frac{1}{a_0}E_0^i(n_0,\vec{n}-\vec{\varepsilon})-\sum_{j=1}^{d}\frac{(-1)^{\varepsilon_j}}{a_j}E_j^i(n_0,\vec{n}-\vec{\varepsilon})$$

$$+\left(1-\frac{1}{\xi}\right)\frac{(-1)^{\varepsilon_i}}{a_i}\{E_0^0(n_0,\vec{n}-\vec{\varepsilon})+\sum_{j=1}^{d}E_j^j(n_0,\vec{n}-\vec{\varepsilon})\}$$

$$=\sum_{\vec{\varepsilon}}\{\frac{1}{a_0}E_0^i(n_0-1,\vec{n}-\vec{\varepsilon})+\sum_{j=1}^{d}\frac{(-1)^{\varepsilon_j}}{a_j}E_j^i(n_0-1,\vec{n}-\vec{\varepsilon})$$

$$-\left(1-\frac{1}{\xi}\right)\frac{(-1)^{\varepsilon_i}}{a_i}\{E_0^0(n_0-1,\vec{n}-\vec{\varepsilon})$$

$$+\sum_{j=1}^{d}E_j^j(n_0-1,\vec{n}-\vec{\varepsilon})\}\}\ .\qquad(4.10)$$

Now we introduce a Fourier transformation,

$$f(n_0,\vec{n})=\frac{1}{M^d}\sum_{\vec{p}=1}^{M}e^{-i\vec{n}\cdot\vec{p}\frac{2\pi}{M}}\tilde{f}(n_0,\vec{p})\ .\qquad(4.11)$$

$f(n_0,\vec{n})$ satisfies a periodic boundary condition, $f(n_0,\vec{n})=f(n_0,\vec{n}+M\vec{\imath})$, where $i=1,2,\cdots,d$ and $\vec{\imath}$ is the unit vector for i-direction. Using eq. (4.11), we transform $A^\mu(n_0,\vec{n}-\vec{\varepsilon})$, $A^\mu(n_0-1,\vec{n}-\vec{\varepsilon})$, $E_\nu^\mu(n_0,\vec{n}-\vec{\varepsilon})$ and $E_\nu^\mu(n_0-1,\vec{n}-\vec{\varepsilon})$ to $\tilde{A}^\mu(n_0,\vec{p})$, $\tilde{A}^\mu(n_0-1,\vec{p})$, $\tilde{E}_\nu^\mu(n_0,\vec{p})$ and $\tilde{E}_\nu^\mu(n_0-1,\vec{p})$. Thus the coupled time evolution equations of $\tilde{A}^\mu(n_0,\vec{p})$ and $\tilde{E}_\nu^\mu(n_0,\vec{p})$ are

$$\frac{1}{2}\tilde{E}_0^\mu(n_0,\vec{p})-\eta_0\tilde{A}^\mu(n_0,\vec{p})$$

$$=-\frac{1}{2}\tilde{E}_0^\mu(n_0-1,\vec{p})-\eta_0\tilde{A}^\mu(n_0-1,\vec{p})\ ,\qquad(4.12)$$

$$\frac{1}{2}\tilde{E}_i^\mu(n_0,\vec{p})-\eta_i\tilde{A}^\mu(n_0,\vec{p})$$

$$=-\frac{1}{2}\tilde{E}_i^\mu(n_0-1,\vec{p})+\eta_0\tilde{A}^\mu(n_0-1,\vec{p})\ ,\qquad(4.13)$$

$$\frac{1}{\xi}\eta_0\tilde{E}_0^0(n_0,\vec{p})-\sum_{j=1}^{d}\eta_j\tilde{E}_j^0(n_0,\vec{p})$$

$$-(1-\frac{1}{\xi})\eta_0\sum_{j=1}^{d}\tilde{E}_j^j(n_0,\vec{p})$$

$$=\frac{1}{\xi}\eta_0\tilde{E}_0^0(n_0-1,\vec{p})+\sum_{j=1}^{d}\eta_j\tilde{E}_j^0(n_0-1,\vec{p})$$

$$-(1-\frac{1}{\xi})\eta_0\sum_{j=1}^{d}\tilde{E}_j^j(n_0-1,\vec{p}),\qquad(4.14)$$

$$\eta_0\tilde{E}_0^i(n_0,\vec{p})-\sum_{j=1}^{d}\eta_j\tilde{E}_j^i(n_0,\vec{p})$$

$$-(1-\frac{1}{\xi})\eta^i\{\tilde{E}_0^0(n_0,\vec{p})+\sum_{j=1}^{d}\tilde{E}_j^j(n_0,\vec{p})$$

$$=\eta_0\tilde{E}_0^i(n_0-1,\vec{p})+\sum_{j=1}^{d}\eta_j\tilde{E}_j^i(n_0-1,\vec{p})$$

$$+(1-\frac{1}{\xi})\eta^i\{\tilde{E}_0^0(n_0-1,\vec{p})+\sum_{j=1}^{d}\tilde{E}_j^j(n_0-1,\vec{p}),\qquad(4.15)$$

where we put

$$\eta_0\equiv\frac{1}{a_0},\ \ \eta_i(\vec{p})\equiv\frac{\sum_{\vec{\varepsilon}}e^{i\vec{\varepsilon}\cdot\vec{p}\frac{2\pi}{M}}\frac{(-1)^{\varepsilon_i}}{a_i}}{\sum_{\vec{\varepsilon}}e^{i\vec{\varepsilon}\cdot\vec{p}\frac{2\pi}{M}}}=\frac{-i\tan p_i\frac{\pi}{M}}{a_i},$$

$$\eta^\mu\equiv\sum_{\nu=0}^{d}g^{\mu\nu}\eta_\nu\,.\qquad(4.16)$$

Eliminating $\tilde{E}_0^0(n_0,\vec{p})$, we obtain the coupled time evolution equations for $\tilde{A}^\mu(n_0,\vec{p})$,

$$\left[\frac{1}{\xi}\eta_0^2-\sum_{j=1}^{d}\eta_j^2\right]\tilde{A}^0(n_0,\vec{p})-(1-\frac{1}{\xi})\eta_0\sum_{j=1}^{d}\eta_j\tilde{A}^j(n_0,\vec{p})$$

$$=\frac{1}{\xi}\eta_0\tilde{E}_0^0(n_0-1,\vec{p})-\left(1-\frac{1}{\xi}\right)\eta_0\sum_{j=1}^{d}\tilde{E}_j^j(n_0-1,\vec{p})$$

$$+\left[\frac{1}{\xi}\eta_0^2+\sum_{j=1}^{d}\eta_j^2\right]\tilde{A}^0(n_0-1,\vec{p})$$

$$+\left(1-\frac{1}{\xi}\right)\eta_0\sum_{j=1}^{d}\eta_j\tilde{A}^j(n_0-1,\vec{p})\,,\qquad(4.17)$$

$$\left[\eta_0^2-\sum_{j=1}^{d}\eta_j^2\right]\tilde{A}^i(n_0,\vec{p})-\left(1-\frac{1}{\xi}\right)\eta^i\eta_0\tilde{A}^0(n_0,\vec{p})$$

$$-\left(1-\frac{1}{\xi}\right)\eta^i\sum_{j=1}^{d}\eta_j\tilde{A}^j(n_0,\vec{p})=\eta_0\tilde{E}_0^i(n_0-1,\vec{p})$$

$$+\left[\eta_0^2+\sum_{j=1}^{d}\eta_j^2\right]\tilde{A}^i(n_0-1,\vec{p})-\left(1-\frac{1}{\xi}\right)\eta^i\eta_0\tilde{A}^0(n_0-1,\vec{p})$$

$$+\left(1-\frac{1}{\xi}\right)\eta^i\sum_{j=1}^{d}\eta_j\tilde{A}^j(n_0-1,\vec{p})\ .\qquad(4.18)$$

We call eqs. (4.17) and (4.18) "the first type coupled equations". We will solve these equations in Sec.7.

4.2. Time evolution equations for A^μ

Next we derive the coupled equations for A^μ only, which are called "the second type time coupled equations". We eliminate all E_ν^μ from eqs. (4.3)-(4.6).

At first, we define the following two operators,

$$\nabla_\mu f(n)=f(n+\hat{\mu})-f(n)\ ,\qquad(4.19)$$

and

$$\widetilde{\nabla}_\mu f(n)=f(n+\hat{\mu})+f(n)\ ,\qquad(4.20)$$

where $\mu=0,1,\cdots,d$ and $n=(n_0,n_1,\cdots,n_d)$. $\hat{\mu}$ is a unit vector in μ-direction. The re1ation,

$$\nabla_\mu\sum_{\varepsilon_\mu}f(n-\varepsilon)=\widetilde{\nabla}_\mu\sum_{\varepsilon_\mu}(-1)^{\varepsilon_\mu}f(n-\varepsilon)\ ,\qquad(4.21)$$

is very useful.

Then, by operating $\prod_{s=0}^{d}\widetilde{\nabla}_s$ on eq. (4.5), we obtain

$$\frac{1}{\xi}(\prod_{s=1}^{d}\widetilde{\nabla}_s)\frac{\nabla_0}{a_0}\sum_{\{\varepsilon\}}E_0^0(n-\varepsilon)$$

$$-\sum_{j=1}^{d}(\prod_{\substack{s=0\\s\neq j}}^{d}\widetilde{\nabla}_s)\frac{\nabla_j}{a_j}\sum_{\{\varepsilon\}}E_j^0(n-\varepsilon)$$

$$-\left(1-\frac{1}{\xi}\right)\sum_{j=1}^{d}(\prod_{s=1}^{d}\widetilde{\nabla}_s)\frac{\nabla_0}{a_0}\sum_{\{\varepsilon\}}E_j^j(n-\varepsilon)\ =0\ .\qquad(4.22)$$

Substituting eqs. (4.3) and (4.4) into eq. (4.22), we have

$$\frac{1}{\xi}(\prod_{s=1}^{d}\tilde{\nabla}_s)\frac{\nabla_0}{a_0}\sum_{\{\varepsilon\}}\frac{(-1)^{\varepsilon_0}}{a_0}A^0(n-\varepsilon)$$

$$-\sum_{j=1}^{d}(\prod_{\substack{s=0\\ s\neq j}}^{d}\tilde{\nabla}_s)\frac{\nabla_j}{a_j}\sum_{\{\varepsilon\}}\frac{(-1)^{\varepsilon_j}}{a_j}A^0(n-\varepsilon)$$

$$-\left(1-\frac{1}{\xi}\right)\sum_{j=1}^{d}(\prod_{\substack{s=1\\ s\neq j}}^{d}\tilde{\nabla}_s)\frac{\nabla_0}{a_0}\frac{\nabla_j}{a_j}\sum_{\{\varepsilon\}}A^j(n-\varepsilon) = 0 \quad . \tag{4.23}$$

On the other hand, eq. (4.6) is rewritten as

$$\sum_{\{\varepsilon\}}\frac{(-1)^{\varepsilon_0}}{a_0}E_0^i(n-\varepsilon)-\sum_{\substack{j=1\\ j\neq i}}^{d}\sum_{\{\varepsilon\}}\frac{(-1)^{\varepsilon_j}}{a_j}E_j^i(n-\varepsilon)$$

$$+\left(1-\frac{1}{\xi}\right)\sum_{\{\varepsilon\}}\frac{(-1)^{\varepsilon_i}}{a_i}E_0^0(n-\varepsilon)-\frac{1}{\xi}\sum_{\{\varepsilon\}}\frac{(-1)^{\varepsilon_i}}{a_i}E_i^i(n-\varepsilon)$$

$$+(1-\frac{1}{\alpha})\sum_{\{\varepsilon\}}\frac{(-1)^{\varepsilon_i}}{a_i}\sum_{\substack{j=1\\ j\neq i}}^{d}E_j^j(n-\varepsilon)\} = 0 \quad .$$

As before, by operating $\prod_{s=0}^{d}\tilde{\nabla}_s$ and using eqs. (4.3) and (4.2), we obtain the result

$$\prod_{s=1}^{d}\tilde{\nabla}_s\frac{\nabla_0}{a_0}\sum_{\{\varepsilon\}}\frac{(-1)^{\varepsilon_0}}{a_0}A^i(n-\varepsilon)$$

$$-\sum_{\substack{j=1\\ j\neq i}}^{d}\prod_{\substack{s=0\\ s\neq j}}^{d}\tilde{\nabla}_s\frac{\nabla_j}{a_j}\sum_{\{\varepsilon\}}\frac{(-1)^{\varepsilon_j}}{a_j}\tilde{A}^i(n-\varepsilon)$$

$$+\left(1-\frac{1}{\xi}\right)\prod_{\substack{s=1\\ s\neq i}}^{d}\tilde{\nabla}_s\frac{\nabla_i}{a_i}\frac{\nabla_0}{a_0}\sum_{\{\varepsilon\}}A^0(n-\varepsilon)$$

$$-\frac{1}{\xi}\prod_{\substack{s=0\\ s\neq i}}^{d}\tilde{\nabla}_s\frac{\nabla_i}{a_i}\sum_{\{\varepsilon\}}\frac{(-1)^{\varepsilon_i}}{a_i}A^i(n-\varepsilon)$$

$$+(1-\frac{1}{\alpha})\sum_{\substack{j=1\\ j\neq i}}^{d}\prod_{\substack{s=0\\ s\neq i,j}}^{d}\tilde{\nabla}_s\frac{\nabla_i}{a_i}\frac{\nabla_j}{a_j}\sum_{\{\varepsilon\}}A^j(n-\varepsilon) = 0 \quad . \tag{4.24}$$

Now eqs. (4.23) and (4.24) are the coupled equations for A^μ only. Using these equations, we will discuss about the gauge symmetry and the dispersion relation.

4.3. Time evolution equations for of $\tilde{\pi}^\mu(n_0,\vec{p})$

The coupled equations of π^μ are formulated on the finite element here. The linear approximation for π^μ is

$$\pi^\mu(x) = \sum_{\{\varepsilon\}}\prod_{s=0}^{d}\left\{\varepsilon_s+(-1)^{\varepsilon_s}\frac{\alpha_s}{a_s}\right\}\pi^\mu(n-\varepsilon) \tag{4.25}$$

as eqs. (4.1) and (4.2). We substitute eq. (4.25) into eq. (3.5) and put $\alpha_s = \frac{a_s}{2}$. Then we obtain

$$\sum_{\{\varepsilon\}}\frac{1}{2}\pi^0(n-\varepsilon) + \frac{1}{\xi}\sum_{\{\varepsilon\}}\frac{(-1)^{\varepsilon_0}}{a_0}A^0(n-\varepsilon)$$

$$+\frac{1}{\xi}\sum_{j=1}^{d}\sum_{\{\varepsilon\}}\frac{(-1)^{\varepsilon_j}}{a_j}A^j(n-\varepsilon)=0 \quad , \tag{4.26}$$

$$\sum_{\{\varepsilon\}}\frac{1}{2}\pi^i(n-\varepsilon)+\sum_{\{\varepsilon\}}\frac{(-1)^{\varepsilon_i}}{a_i}A^0(n-\varepsilon)$$

$$+\sum_{\{\varepsilon\}}\frac{(-1)^{\varepsilon_0}}{a_0}A^i(n-\varepsilon)=0 \quad . \tag{4.27}$$

The terms which are defined at time n_0 and n_0-1 are separated such as

$$\sum_{\vec{\varepsilon}}\frac{1}{2}\pi^0(n_0,\vec{n}-\vec{\varepsilon})+\frac{1}{\xi}\sum_{\vec{\varepsilon}}\frac{1}{a_0}A^0(n_0,\vec{n}-\vec{\varepsilon})$$

$$+\frac{1}{\xi}\sum_{j=1}^{d}\sum_{\vec{\varepsilon}}\frac{(-1)^{\varepsilon_j}}{a_j}A^j(n_0,\vec{n}-\vec{\varepsilon})$$

$$=-\sum_{\vec{\varepsilon}}\frac{1}{2}\pi^0(n_0-1,\vec{n}-\vec{\varepsilon})+\frac{1}{\xi}\sum_{\vec{\varepsilon}}\frac{1}{a_0}A^0(n_0-1,\vec{n}-\vec{\varepsilon})$$

$$-\frac{1}{\xi}\sum_{j=1}^{d}\sum_{\vec{\varepsilon}}\frac{(-1)^{\varepsilon_j}}{a_j}A^j(n_0-1,\vec{n}-\vec{\varepsilon}) \quad , \tag{4.28}$$

$$\sum_{\vec{\varepsilon}}\frac{1}{2}\pi^i(n_0,\vec{n}-\vec{\varepsilon})+\sum_{\vec{\varepsilon}}\frac{(-1)^{\varepsilon_i}}{a_i}A^0(n_0,\vec{n}-\vec{\varepsilon})$$

$$+\sum_{\vec{\varepsilon}}\frac{1}{a_0}A^i(n_0,\vec{n}-\vec{\varepsilon}) = -\sum_{\vec{\varepsilon}}\frac{1}{2}\pi^i(n_0-1,\vec{n}-\vec{\varepsilon})$$

$$-\sum_{\vec{\varepsilon}}\frac{(-1)^{\varepsilon_i}}{a_i}A^0(n_0-1,\vec{n}-\vec{\varepsilon})+\sum_{\vec{\varepsilon}}\frac{1}{a_0}A^i(n_0-1,\vec{n}-\vec{\varepsilon}) \quad . \tag{4.29}$$

They are Fourier-transformed by eq. (4.11) into

$$\frac{1}{2}\tilde{\pi}^0(n_0,\vec{p})+\frac{1}{\xi}\sum_{\mu=0}^{d}\eta_\mu\tilde{A}^\mu(n_0,\vec{p})$$

$$= -\frac{1}{2}\tilde{\pi}^0(n_0-1,\vec{p})+\frac{1}{\xi}\sum_{\mu=0}^{d}g^{\mu\mu}\eta_\mu\tilde{A}^\mu(n_0-1,\vec{p}) \quad . \tag{4.30}$$

$$\frac{1}{2}\tilde{\pi}^i(n_0,\vec{p})-\eta^i\tilde{A}^0(n_0,\vec{p})+\eta_0\tilde{A}^i(n_0,\vec{p})$$

$$= -\frac{1}{2}\tilde{\pi}^i(n_0-1,\vec{p})+\eta^i\tilde{A}^0(n_0-1,\vec{p})$$

$$+\eta_0\tilde{A}^i(n_0-1,\vec{p}) \quad . \tag{4.31}$$

Equations (4.30) and (4.31) are the coupled equations of $\tilde{\pi}^\mu(n_0,\vec{p})$.

4.4. Relations among $\tilde{A}^\mu(n_0,\vec{p})$, $\tilde{\pi}^\mu(n_0,\vec{p})$ and $\tilde{E}^u_v(n_0,\vec{p})$

We have defined $\tilde{E}_\nu^\mu(x)$ and $\tilde{\pi}^\mu(x)$ by eqs. (3.3) and (3.5) respectively. Here, according to the prescription mentioned in Sec. 2, we construct relations among $\tilde{A}^\mu(n_0,\vec{p})$, $\tilde{\pi}^\mu(n_0,\vec{p})$ and $\tilde{E}_\nu^\mu(n_0,\vec{p})$. As we have shown already, we get the relation between $E_0^\mu(n)$ and $A^\mu(n)$, eq. (4.3), from eq. (3.3). On the other hand, substituting eq.(3.3) into eq. (3.5), we have

$$\pi^0(x) = -\frac{1}{\xi}\sum_{\mu=0}^{d}E_\mu^\mu(x) \tag{4.32}$$

and

$$\pi^i(x) = -E_i^0(x)-E_0^i(x) \quad . \tag{4.33}$$

Substituting eqs. (4.2) and (4.25) into eqs. (4.32) and (4.33), we obtain

$$\sum_{\{\varepsilon\}}\pi^0(n-\varepsilon) = -\frac{1}{\xi}\sum_{\mu=0}^{d}\sum_{\{\varepsilon\}}E_\mu^\mu(n-\varepsilon) \tag{4.34}$$

and

$$\sum_{\{\varepsilon\}}\pi^i(n-\varepsilon) = -\sum_{\{\varepsilon\}}E_i^0(n-\varepsilon)-\sum_{\{\varepsilon\}}E_0^i(n-\varepsilon) \quad . \tag{4.35}$$

In the case of the continuum theory, the relations among $A^\mu(x)$, $\pi^\mu(x)$, and $E_\nu^\mu(x)$ are defined locally. However, they are defined in the form of recursion relations on the finite element. In order to get relations among these variables at any fixed time directly, we use the technique of the Fourier transformation. Under the transformation as

$$f(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\omega\frac{1}{M^d}\sum_{\vec{p}=1}^{M}e^{-i\omega n_0-i\vec{p}\cdot\vec{n}\frac{2\pi}{M}}\tilde{f}(\omega,\vec{p})$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\omega\frac{1}{M^d}\sum_{P=1}^{M}e^{-in\cdot P}\tilde{f}(P)\ , \tag{4.36}$$

where $P_0\equiv\omega_0$, $P_i\equiv p_i\frac{2\pi}{M}$, eqs. (4.3), (4.34), and (4. 35) are transformed into

$$\tilde{E}_\nu^\mu(P)=2\xi_\nu(P)\tilde{A}^\mu(P)\ , \tag{4.37}$$

$$\tilde{\pi}^0(P)=-\frac{1}{\xi}\sum_{\mu=0}^{d}\tilde{E}_\mu^\mu(P)\ , \tag{4.38}$$

$$\tilde{\pi}^i(P)=-\tilde{E}_i^0(P)-\tilde{E}_0^i(P)\ , \tag{4.39}$$

where

$$\xi_\nu(P)\equiv\frac{\sum_{\varepsilon_\nu}e^{i\varepsilon_\nu P_\nu}\frac{(-1)^{\varepsilon_\nu}}{a_\nu}}{\sum_{\varepsilon_\nu}e^{i\varepsilon_\nu P_\nu}}=\frac{-i\tan\frac{P_\nu}{2}}{a_\nu}\ ,\ \xi^\mu=\sum_{\nu=0}^{d}g^{\mu\nu}\xi_\nu\ . \tag{4.40}$$

$\tilde{f}(P)$ in eq. (4.36) is related to $\tilde{f}(n_0,\vec{p})$ in eq. (4.11) by

$$\tilde{f}(n_0,\vec{p})\ =\frac{1}{2\pi}\int_{-\pi}^{\pi}d\omega\, e^{-i\omega n_0}\tilde{f}(P)\quad . \tag{4.41}$$

Applying eq. (4.41) to eqs. (4.37), (4.38) and (4.39), we get

$$\tilde{E}_i^\mu(n_0,\vec{p})=2\xi_i(\vec{p})\tilde{A}^\mu(n_0,\vec{p})\ , \tag{4.42}$$

$$\tilde{\pi}^0(n_0,\vec{p})=-\frac{1}{\xi}\sum_{\mu=0}^{d}\tilde{E}_\mu^\mu(n_0,\vec{p})\ , \tag{4.43}$$

$$\tilde{\pi}^i(n_0,\vec{p})=-\tilde{E}_i^0(n_0,\vec{p})-\tilde{E}_0^i(n_0,\vec{p})\ . \tag{4.44}$$

Notice that the difference between the variables $\vec{p}$ and $P=(P_0,\vec{p})$ which are define at the below of eq. (4. 36).

4.5. Additive note

Both types of the coupled equations are equivalent to each other. We see how the field operators at any time are determined from the ones at an initial time in the both cases.

i) The first type coupled equations:

We express $\tilde{A}^\mu(n_0,\vec{p})$, $\tilde{\pi}^\mu(n_0,\vec{p})$ and $\tilde{E}_\nu^\mu(n_0,\vec{p})$ in terms of the same quantities at the previous time n_0-1. We can show $\tilde{A}^\mu(n_0,\vec{p})$ in terms of $\tilde{A}^\mu(n_0-1,\vec{p})$ and $\tilde{E}_\nu^\mu(n_0-1,\vec{p})$ using the first type coupled equations (4.17) and (4.18). Then $\tilde{\pi}^0(n_0,\vec{p})$ is known from $\tilde{A}^\mu(n_0,\vec{p})$, $\tilde{A}^\mu(n_0-1,\vec{p})$ and $\tilde{\pi}^0(n_0-1,\vec{p})$. Once $\tilde{A}^\mu(n_0,\vec{p})$ and $\tilde{\pi}^\mu(n_0,\vec{p})$ are known, the coupled equations (4.42), (4.43) and (4.44) are solved for $\tilde{E}_\nu^\mu(n_0,\vec{p})$. Thus, it follows that $\tilde{A}^\mu(n_0,\vec{p})$, $\tilde{\pi}^\mu(n_0,\vec{p})$ and $\tilde{E}_\nu^\mu(n_0,\vec{p})$ are determined from $\tilde{A}^\mu(n_0-1,\vec{p})$, $\tilde{\pi}^\mu(n_0-1,\vec{p})$ and $\tilde{E}_\nu^\mu(n_0-1,\vec{p})$. By iterating these steps, we obtain $\tilde{A}^\mu(n_0,\vec{p})$, $\tilde{\pi}^\mu(n_0,\vec{p})$ and $\tilde{E}_\nu^\mu(n_0,\vec{p})$ at any time.

ii) The second type coupled equations:

Because eqs. (4.23) and (4.24) contain $A^\mu(n_0+1)$, $A^\mu(n_0)$ and $A^\mu(n_0-1)$, $A^\mu(n_0+1)$ is found once $A^\mu(n_0)$ and $A^\mu(n_0-1)$ are known. Originally, the equation of motion for $A^\mu(x)$ is the second order differential equation. When we solved the equation regarding as the time evolution equation, $A^\mu(x)$ and $\partial_0A^\mu(x)$ at initial time are needed. It means that $A^\mu(n_0)$ and $A^\mu(n_0-1)$ are needed in the method of finite element. Next, if $\pi^\mu(n_0)$ is given, $\pi^\mu(n_0+1)$ is known from $\pi^\mu(n_0)$, $A^\mu(n_0+1)$ and $A^\mu(n_0)$ through eqs. (4.28) and (4.29). Thus $A^\mu(n_0+1)$ and $\pi^\mu(n_0+1)$ are determined by the initial condition$A^\mu(n_0)$, $A^\mu(n_0-1)$ and $\pi^\mu(n_0)$. By the iterations, we obtain $A^\mu(n_0)$ and $\pi^\mu(n_0)$ at any time.

5. The dispersion relations

For the gauge field, there is no doubling problem in the standard lattice gauge theory. Here, to make sure, we investigate the dispersion relation of the U(1) gauge field.

At first, in the continuum case, eq.(3.2) is transformed into

$$-p^2\tilde{A}^\mu(p)+\left(1-\frac{1}{\xi}\right)p^\mu\sum_{\nu=0}^{d}p_\nu\tilde{A}^\nu(p)=0 \tag{5.1}$$

by

$$A^\mu(x)=\frac{1}{(2\pi)^{d+1}}\int d^{d+1}p\tilde{A}^\mu(p)e^{-ip\cdot x}. \tag{5.2}$$

From eq. (5.1), the dispersion relation,

$$p^2=0, \tag{5.3}$$

is obtained.

In the case of the U(1) gauge field on the finite element, using the Fourier transformation eq.(4.36), we get directly

$$\left[\frac{1}{\xi}\xi_0^2-\sum_{j=1}^{d}\xi_j^2\right]\tilde{A}^0(\omega,\vec{p})-\left(1-\frac{1}{\xi}\right)\sum_{j=1}^{d}\xi_0\xi_j\tilde{A}^j(\omega,\vec{p})=0 \tag{5.4}$$

and

$$\left[\xi_0^2-\sum_{\substack{j=1\\ j\neq i}}^{d}\xi_j^2\right]\tilde{A}^i(\omega,\vec{p})-\frac{1}{\alpha}\xi_i^2\tilde{A}^i(\omega,\vec{p})$$
$$-\left(1-\frac{1}{\alpha}\right)\xi^i\sum_{\substack{\mu=1\\ \mu\neq i}}^{d}\xi_\mu\tilde{A}^\mu(\omega,\vec{p})=0 \tag{5.5}$$

from eqs.(4.23) and (4.24) respectively, where ξ_μ is defined by eq.(4.40). Eqs. (5.4) and (5.5) are collected into the form of

$$\left[\xi_0^2 - \sum_{j=1}^{d} \xi_j^2\right]\tilde{A}^\mu(\omega,\vec{p}) - \left(1-\frac{1}{\xi}\right)\sum_{\rho=0}^{d} g^{\mu\rho}\xi_\rho \sum_{\nu=0}^{d}\xi_\nu \tilde{A}^\nu(\omega,\vec{p}) = 0 \quad . (5.6)$$

From eq (5.6), the dispersion relation of the U(1.) gauge field on the finite element,

$$\xi_0^2 - \sum_{j=1}^{d}\xi_j^2 = 0 \qquad , \qquad (5.7)$$

is obtained. In the naive continuum limit, eq. (5.7) leads eq. (5.3).

6. The gauge symmetry of the U(l) gauge field on the finite and the necessity for the gauge fixing term

In constructing the U(l) gauge field on the finite element, we have started from the equation of motion with the gauge fixing term in the continuum theory as eq.(3.2). If we put $\frac{1}{\xi}=0$, the corresponding continuum theory is gauge symmetric. Then the symmetry should be reflected in the U(l) gauge field theory on the finite element. We construct the gauge transformation on a finite element and verify that the equation of motion is invariant under its transformation using the second type equations of motion (4.23) and (4.24).

In the continuum theory, the equation of motion eq.(3.2) with $\frac{1}{\xi}=0$ is invariant under the gauge transformation,

$$A^{\mu\prime}(x) = A^\mu(x) + \frac{i}{g}\partial^\mu\theta(x) \quad , \qquad (6.1)$$

where g is the U(l) coupling constant and $\theta(x)$ is a local gauge function. Substituting eq.(4.1) for $\theta(x)$

$$\theta(x) = \sum_{\{\varepsilon\}}\prod_{s=0}^{d}\left\{\varepsilon_s + (-1)^{\varepsilon_s}\frac{\alpha_s}{a_s}\right\}\theta(n-\varepsilon) \qquad (6.2)$$

into Eq.(6.1), and putting $\alpha_s = \frac{a_s}{2}$, we have

$$\sum_{\{\varepsilon\}} A^{\mu\prime}(n-\varepsilon) = \sum_{\{\varepsilon\}} A^\mu(n-\varepsilon) + \frac{i}{g}\sum_{\nu=0}^{d} g^{\mu\nu}\sum_{\{\varepsilon\}}\frac{2(-1)^{\varepsilon_\nu}}{a_\nu}\theta(n-\varepsilon) \qquad (6.3)$$

which is the gauge transformation of the U(1) field on the finite element.

Next we show that eqs. (4.23) and (4.24) are invariant under eq.(6.3). The invariance may be shown in the momentum space as well as the coordinate space. Here we work in the momentum space. By using eq.(4.36), eq.(6.3) is transformed into

$$\tilde{A}^\mu(\mathrm{P})' = \tilde{A}^\mu(\mathrm{P}) + \frac{2i}{g}\xi^\mu\tilde{\theta}(P) \quad . \qquad (6.4)$$

On the other hand, the equation of motion with the gauge fixing term on the finite element in the momentum space is already derived as eq.(5.6). From eqs.(5.6) and (6.3), we have

$$\sum_{\nu=0}^{d}\xi_\nu\xi^\nu\tilde{A}^{\mu\prime}(\omega,\vec{p}) - \left(1-\frac{1}{\xi}\right)\xi^\mu\sum_{\nu=0}^{d}\xi_\nu\tilde{A}^{\nu\prime}(\omega,\vec{p})$$

$$= \frac{1}{\xi}\sum_{\nu=0}^{d}\xi_\nu\xi^\nu\frac{2i}{g}\xi^\mu\tilde{\theta}(P) \quad .$$

Thus, if there is not the gauge fixing term, the equation of motion (5.6) is invariant under the gauge transformation eq.(6.4).

The reason why we have the gauge fixing term is the same as the one in the continuum theory. From eqs.(4.17) and (4.18), setting $\frac{1}{\xi}=0$, the equations of motion without the fixing term,

$$-\sum_{j=1}^{d}\eta_j^2\tilde{A}^0(n_0,\vec{p}) - \eta_0\sum_{j=1}^{d}\eta_j\tilde{A}^j(n_0,\vec{p})$$
$$= -\eta_0\sum_{j=1}^{d}\tilde{E}_j^j(n_0-1,\vec{p}) + \sum_{j=1}^{d}\eta_j^2\tilde{A}^0(n_0-1,\vec{p})$$
$$+\eta_0\sum_{j=1}^{d}\eta_j\tilde{A}^j(n_0-1,\vec{p}) \qquad (6.5)$$

and

$$\left[\eta_0^2 - \sum_{j=1}^{d}\eta_j^2\right]\tilde{A}^i(n_0,\vec{p}) - \eta^i\eta_0\tilde{A}^0(n_0,\vec{p}) - \eta^i\sum_{j=1}^{d}\eta_j\tilde{A}^j(n_0,\vec{p})$$
$$= \eta_0\tilde{E}_0^i(n_0-1,\vec{p}) + \left[\eta_0^2 + \sum_{j=1}^{d}\eta_j^2\right]\tilde{A}^i(n_0-1,\vec{p})$$
$$-\eta^i\eta_0\tilde{A}^0(n_0-1,\vec{p}) + \eta^i\sum_{j=1}^{d}\eta_j\tilde{A}^j(n_0-1,\vec{p}) \qquad (6.6)$$

are obtained. The quations (6.5) and (6.6) are brought together in the more compact form,

$$\sum_{\nu=0}^{d} I^{\mu\nu}\tilde{A}_\nu(n_0,\vec{p}) = J^\mu(n_0-1,\vec{p}) \quad , \qquad (6.7)$$

where J^0 and J^i denote the right hand side of eqs. (6.5) and (6.6) respectively and

$$I_\nu^\mu = \sum_{\rho=0}^{d}\eta_\rho\eta^\rho g_\nu^\mu - \eta^\mu\eta_\nu \quad . \qquad (6.8)$$

If we can invert the matrix (I_ν^μ), eq.(6.7) is solved. However, $\det(I_\nu^\mu)=0$ and $(I_\nu^\mu)^{-1}$ does not exist. This is the same situation as the operator $\sum_{\rho=0}^{d}\partial_\rho\partial^\rho g_\nu^\mu - \partial^\mu\partial_\nu$ in eq.(3.21). Thus we must require the gauge fixing term.

7. The one-time-step solutions

By solving the first type coupled equations as eqs. (4.17) and (4.18), we obtain $\tilde{A}^\mu(n_0,\vec{p})$. Further we get also $\tilde{\pi}^\mu(n_0,\vec{p})$ from eqs. (4.30) and (4.31).

Multiplying eq. (4.18) by η_i and summing up for index i, we get

$$\sum_{i=1}^d \eta_i \tilde{A}^i(n_0,\vec{p}) = \frac{\eta_0}{\eta_0^2-\frac{1}{\xi}\sum_{j=1}^d \eta_j^2}\sum_{i=1}^d \eta_i \tilde{E}_0^i(n_0-1,\vec{p})$$

$$+\frac{\eta_0^2+\frac{1}{\xi}\sum_{j=1}^d \eta_j^2}{\eta_0^2-\frac{1}{\xi}\sum_{j=1}^d \eta_j^2}\sum_{i=1}^d \eta_i \tilde{A}^i(n_0-1,\vec{p}) + \left(1-\frac{1}{\xi}\right)\frac{\sum_{i=1}^d \eta_i^2\eta_0}{\eta_0^2-\frac{1}{\xi}\sum_{j=1}^d \eta_j^2}\tilde{A}^0(n_0-1,\vec{p})$$

$$-\left(1-\frac{1}{\xi}\right)\frac{\sum_{i=1}^d \eta_i^2\eta_0}{\eta_0^2-\frac{1}{\xi}\sum_{j=1}^d \eta_j^2}\tilde{A}^0(n_0,\vec{p}) \quad . \tag{7.1}$$

Substituting eq. (7.1) into Eq. (4.17) and solving for $\tilde{A}^0(n_0,\vec{p})$, we obtain

$$\tilde{A}^0(n_0,\vec{p}) = \frac{\eta_0^2-\frac{1}{\xi}\sum_{j=1}^d \eta_j^2}{(\eta_0^2-\sum_{j=1}^d \eta_j^2)^2}\ \eta_0\tilde{E}_0^0(n_0-1,\vec{p})$$

$$+\frac{(\xi-1)\eta_0^2}{(\eta_0^2-\sum_{j=1}^d \eta_j^2)^2}\sum_{i=1}^d \eta_i \tilde{E}_0^i(n_0-1,\vec{p})$$

$$-(\xi-1)\frac{\eta_0^2-\frac{1}{\xi}\sum_{j=1}^d \eta_j^2}{\left(\eta_0^2-\sum_{j=1}^d \eta_j^2\right)^2}\eta_0\sum_{i=1}^d \tilde{E}_i^i(n_0-1,\vec{p})$$

$$+\ \frac{\eta_0^4+2(\xi-1)\eta_0^2\sum_{j=1}^d \eta_j^2-(\sum_{j=1}^d \eta_j^2)^2}{\left(\eta_0^2-\sum_{j=1}^d \eta_j^2\right)^2}\tilde{A}^0(n_0-1,\vec{p})$$

$$+\ \frac{2(\xi-1)\eta_0^3}{\left(\eta_0^2-\sum_{j=1}^d \eta_j^2\right)^2}\sum_{i=1}^d \eta_i \tilde{A}^i(n_0-1,\vec{p}) \quad . \tag{7.2}$$

Substituting eqs. (7.1) and (7.2) into eq. (4.18), we have

$$\tilde{A}^i(n_0,\vec{p}) = \frac{\left(1-\frac{1}{\xi}\right)\eta^i\eta_0^2}{\left(\eta_0^2-\sum_{j=1}^d \eta_j^2\right)^2}\ \tilde{E}_0^0(n_0-1,\vec{p})$$

$$+\frac{\eta_0}{\eta_0^2-\sum_{j=1}^d \eta_j^2}\ \tilde{E}_0^i(n_0-1,\vec{p})$$

$$+\frac{(\xi-1)\eta^i\eta_0}{\left(\eta_0^2-\sum_{j=1}^d \eta_j^2\right)^2}\ \sum_{j=1}^d \eta_j \tilde{E}_0^j(n_0-1,\vec{p})$$

$$-\frac{(\xi-1)\left(1-\frac{1}{\xi}\right)\eta^i\eta_0^2}{\left(\eta_0^2-\sum_{j=1}^d \eta_j^2\right)^2}\ \sum_{j=1}^d \tilde{E}_j^j(n_0-1,\vec{p})$$

$$+\frac{2(\xi-1)\eta^i\eta_0\sum_{i=1}^d \eta_i^2}{\left(\eta_0^2-\sum_{j=1}^d \eta_j^2\right)^2}\tilde{A}^0(n_0-1,\vec{p})$$

$$+\frac{\eta_0^2+\sum_{i=1}^d \eta_i^2}{\eta_0^2-\sum_{j=1}^d \eta_j^2}\tilde{A}^i(n_0-1,\vec{p})$$

$$+\frac{2(\xi-1)\eta^i\eta_0^2}{\left(\eta_0^2-\sum_{j=1}^d \eta_j^2\right)^2}\ \sum_{j=1}^d \eta_j \tilde{A}^j(n_0-1,\vec{p}) \quad . \tag{7.3}$$

Next eliminating $\tilde{A}^i(n_0,\vec{p})$ in eq.(4.30) by eq.(7.1) and further eliminating $\tilde{A}^0(n_0,\vec{p})$ in its result by eq. (7.2), we get

$$\tilde{\pi}^0(n_0,\vec{p}) = -\tilde{\pi}^0(n_0-1,\vec{p}) - \frac{2}{\xi}\frac{2}{\eta_0^2-\sum_{j=1}^d \eta_j^2}\ \tilde{E}_0^0(n_0-1,\vec{p})$$

$$-2\frac{\eta_0}{\eta_0^2-\sum_{j=1}^d \eta_j^2}\sum_{i=1}^d \eta_i \tilde{E}_0^i(n_0-1,\vec{p})$$

$$+2\left(1-\frac{1}{\xi}\right)\frac{\eta_0}{\eta_0^2-\sum_{j=1}^d \eta_j^2}\sum_{i=1}^d \tilde{E}_i^i(n_0-1,\vec{p})$$

$$-4\frac{\eta_0\sum_{i=1}^d \eta_i^2}{\eta_0^2-\sum_{j=1}^d \eta_j^2}\ \tilde{A}_0(n_0-1,\vec{p})$$

$$-4\frac{\eta_0^2}{\eta_0^2-\sum_{j=1}^d \eta_j^2}\sum_{i=1}^d \eta_i \tilde{A}^i(n_0-1,\vec{p}) \quad . \tag{7.4}$$

Substituting eqs.(7.2) and (7.3) into eq.(4.31), we have

$$\tilde{\pi}^i(n_0,\vec{p}) = -\tilde{\pi}^i(n_0-1,\vec{p}) + \frac{2}{\xi}\frac{\eta^i\eta_0}{\eta_0^2-\sum_{j=1}^d \eta_j^2}\ \tilde{E}_0^0(n_0-1,\vec{p})$$

$$-2\frac{\eta_0^2}{\eta_0^2-\sum_{j=1}^d \eta_j^2}\tilde{E}_0^i(n_0-1,\vec{p})$$

$$-2\left(1-\frac{1}{\xi}\right)\frac{\eta^i\eta_0}{\eta_0^2-\sum_{j=1}^d \eta_j^2}\sum_{j=1}^d \tilde{E}_j^j(n_0-1,\vec{p})$$

$$+4\frac{\eta^i\eta_0}{\eta_0^2-\sum_{j=1}^d \eta_j^2}\tilde{A}_0(n_0-1,\vec{p})$$

$$-4\frac{\eta_0}{\eta_0^2-\sum_{j=1}^d \eta_j^2}\tilde{A}^i(n_0-1,\vec{p}) \quad . \tag{7.5}$$

The one-time-step solutions of the field equation are given by eqs.(3.8) – (3.10). As mentioned Sec.4, $\tilde{E}_\nu^\mu(n_0,\vec{p})$ is determined automatically by solving eqs.(4.37) – (4.39). By iterating these procedures, we can obtain the field operators at any time.

8. The equal-time commutation relations on the finite element and the consistency of quantization

We quantize canonically the U(1) gauge field on the finite element. In this system, the dynamical variable is $A^\mu(n_0,\vec{n})$ and its conjugate momentum $\pi^\mu(n_0,\vec{n})$. We demand equal-time commutation relations,

$$[A^\rho(n_0,\vec{n}),\pi^\nu(n_0,\vec{m})] = ig^{\rho\nu}\delta_{\vec{n},\vec{m}}^{(d)}\frac{1}{V}\ , \tag{8.1}$$

$$[A^\rho(n_0,\vec{n}),A^\nu(n_0,\vec{m})] = 0\ , \tag{8.2}$$

$$[\pi^\rho(n_0,\vec{n}),\pi^\nu(n_0,\vec{m})] = 0\ , \tag{8.3}$$

where V is the volume of the d-dimensional parallelepiped and $\delta_{\vec{n},\vec{m}}^{(d)}$ is the d-dimensional Kronecker δ. The Fourier transformed version of these relations are

$$\left[\tilde{A}^\rho(n_0,\vec{p}),\tilde{\pi}^\nu(n_0,\vec{k})\right] = ig^{\rho\nu}\delta_{\vec{p},-\vec{k}}^{(d)}\frac{M^d}{V}\ , \tag{8.4}$$

$$\left[\tilde{A}^\rho(n_0,\vec{p}),\tilde{A}^\nu(n_0,\vec{k})\right] = 0\ , \tag{8.5}$$

$$\left[\tilde{\pi}^\rho(n_0,\vec{p}),\tilde{\pi}^\nu(n_0,\vec{k})\right] = 0\ , \tag{8.6}$$

by using eq.(4.11).

Then, we must ask whether the equal-time commutation relations are consistent with the field equations on the finite element or not. Thus, we must prove the following consistency condition: When the equal-time commutation relations are given at initial time, the field operators at arbitrary time which are developed according to the field equations must satisfy the same relations.

This is one of major problems in constructing a quantum field theory on the finite element. The proof for the U(1) gauge field to satisfy the condition will be given in Sec.9.

Since the first type coupled field equations and these one-time-step solution contain $\tilde{E}^{\mu}_{\nu}(n_0,\vec{p})$, it is convenient to derive the equal-time commutation relations between $\tilde{A}^{\rho}(n_0,\vec{p})$ and $\tilde{E}^{\mu}_{\nu}(n_0,\vec{k})$, between $\tilde{\pi}^{\rho}(n_0,\vec{p})$ and $\tilde{E}^{\mu}_{\nu}(n_0,\vec{k})$ and between $\tilde{E}^{\mu}_{\nu}(n_0,\vec{p})$'s themselves. These relations are obtained from eqs. (8.4)–(8.6) using eqs. (4.42) – (4.44). The results are

$$\left[\tilde{A}^{\rho}(n_0,\vec{p}),\ \tilde{E}^{0}_{0}(n_0,\vec{k})\right] = -i\xi g^{\rho 0}\delta^{(d)}_{\vec{p},-\vec{k}}\frac{M^d}{V}\ , \qquad (8.7a)$$

$$\left[\tilde{A}^{\rho}(n_0,\vec{p}),\ \tilde{E}^{i}_{0}(n_0,\vec{k})\right] = -ig^{\rho i}\delta^{(d)}_{\vec{p},-\vec{k}}\frac{M^d}{V}\ , \qquad (8.7b)$$

$$\left[\tilde{A}^{\rho}(n_0,\vec{p}),\ \tilde{E}^{\nu}_{i}(n_0,\vec{k})\right] = 0\ , \qquad (8.7c)$$

for $\tilde{A}^{\rho}$ and $\tilde{E}^{\nu}_{\mu}$,

$$\left[\tilde{E}^{0}_{0}(n_0,\vec{p}),\ \tilde{\pi}^{0}(n_0,\vec{k})\right] = 0\ , \qquad (8.8a)$$

$$\left[\tilde{E}^{0}_{0}(n_0,\vec{p}),\ \tilde{\pi}^{i}(n_0,\vec{k})\right] = 2i\eta_i(\vec{p})\delta^{(d)}_{\vec{p},-\vec{k}}\frac{M^d}{V}\ , \qquad (8.8b)$$

$$\left[\tilde{E}^{i}_{0}(n_0,\vec{p}),\ \tilde{\pi}^{\nu}(n_0,\vec{k})\right] = -2ig^{0\nu}\eta_i(\vec{p})\delta^{(d)}_{\vec{p},-\vec{k}}\frac{M^d}{V}\ , \qquad (8.8c)$$

$$\left[\tilde{E}^{\rho}_{i}(n_0,\vec{p}),\ \tilde{\pi}^{\nu}(n_0,\vec{k})\right] = 2ig^{\rho\nu}\eta_i(\vec{p})\delta^{(d)}_{\vec{p},-\vec{k}}\frac{M^d}{V}\ , \qquad (8.8d)$$

for E^{ρ}_{μ} and π^{ν}, and

$$\left[\tilde{E}^{0}_{0}(n_0,\vec{p}),\ \tilde{E}^{0}_{0}(n_0,\vec{k})\right] = 0\ , \qquad (8.9a)$$

$$\left[\tilde{E}^{0}_{0}(n_0,\vec{p}),\ \tilde{E}^{i}_{0}(n_0,\vec{k})\right] = 2i(\xi-1)\eta_i(\vec{p})\delta^{(d)}_{\vec{p},-\vec{k}}\frac{M^d}{V}\ , \qquad (8.9b)$$

$$\left[\tilde{E}^{i}_{0}(n_0,\vec{p}),\ \tilde{E}^{j}_{0}(n_0,\vec{k})\right] = 0\ , \qquad (8.9c)$$

$$\left[\tilde{E}^{\rho}_{i}(n_0,\vec{p}),\ \tilde{E}^{0}_{0}(n_0,\vec{k})\right] = -2i\xi g^{\rho 0}\eta_i(\vec{p})\delta^{(d)}_{\vec{p},-\vec{k}}\frac{M^d}{V}\ , \qquad (8.9d)$$

$$\left[\tilde{E}^{\rho}_{j}(n_0,\vec{p}),\ \tilde{E}^{i}_{0}(n_0,\vec{k})\right] = -2i\xi g^{\rho i}\eta_j(\vec{p})\delta^{(d)}_{\vec{p},-\vec{k}}\frac{M^d}{V}\ , \qquad (8.9e)$$

$$\left[\tilde{E}^{\rho}_{i}(n_0,\vec{p}),\ \tilde{E}^{\nu}_{j}(n_0,\vec{k})\right] = 0\ , \qquad (8.9f)$$

for $\tilde{E}^{\rho}_{\mu}$'s. Notice the correspondence with the equal-time commutation relations eqs.(3.8) – (3.10) in the continuum theory.

9. The proof of the consistency on quantization

It is proven by direct calculations that the canonically quantized system of the U(1) gauge field on the finite element satisfies the consistency condition on the quantization. Those calculations are direct but very lengthy and tedious. We will restrict ourselves to giving an outline here.

At initial time n_0-1, we require the equal-time commutation relations (8.1) –(8.3). Then eqs.(8.7) – (8.9) at time n_0-1 hold automatically as shown in Sec.8. Using these relations and the one-time-step solutions eqs.(7.2) –-(7.5), we confirm the equal-time commutation relations (8.1) –(8.3) to hold at time n_0 and find that the gauge dependence disappears. Then eqs. (8.7) –-(8.9) at time n_0 hold automatically. Thus the equal-time commutation relations (8.1) –(8.3) and (8.7) –-(8.9) are preserved in developing one-time-step. By iterating this procedure, it follows that the equal-time commutation relations at any time is kept to be invariant.

10. Conclusion and discussion

The results obtained in this paper are the followings. The two types of field equations, which were equivalent each other, were derived. Using the second type equations, we exhibited the dispersion relation which has the desirable continuum limit, and further, showed that the equations have the gauge symmetry on the finite element which was constructed by applying the method of the finite element to the continuum case directly. Next, the first type field equation was solved and using its solution, we proved directly that the consistency condition is satisfied.

The remarkable point of our formulation for U(1) gauge field is that the gauge field space is non-compact and the gauge fixing is needed. In a sense, it might be said that this formulation is more nearer to the continuum theory than the standard lattice gauge theory with respect to the properties of the theory.

The coupling to a matter field is very important. An investigation of the matter coupled gauge theory in (d+1)-dimensional space-time is now in progress.

Finally we also should extend the formalism to the non-Abelian case. The interesting attempt has been done but still now it is not satisfactory because these formalism include the link variable.[10] The more desirable formalism should be formulated as non-compact form. This is an important future problem.

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【日本語要旨】

Quantized Pure U (1) Gauge Field Theory on Finite Element in (d+1)-Dimensional Space-Time

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(d+1) 次元時空間の有限要素上で、純なU(1)群をゲージ対称性に持つ場の量子理論を構成する。場の方程式を有限要素上に定式化し、それらが適正な分散関係を持つことを示す。さらに有限要素上の場の方程式が連続時空間上のゲージ対称性に相当する対称性を保持し、ゲージ固定が必要であることを示す。さらに、場の方程式を解くことによって、量子化に関する無矛盾性を証明する。