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On the highest Lyubeznik number

By KEN-ICHIROH KAWASAKI

Department of Mathematics, Nara University of Education, Takabatake-cho, Nara, 630-8528, Japan.
e-mail: kawaken@nara-edu.ac.jp

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Abstract

In this paper, we will prove that the highest Lyubeznik number $\lambda_{d,d}(A)$ is one if A is a Cohen–Macaualy local ring containing a field, where d is the dimension of A.

We assume that all rings are commutative and noetherian with identity throughout this paper.

1. *Introduction*

In this paper, we shall prove:

THEOREM 1. *Let A be a local ring containing a field with dimension d. If A is Cohen–Macaualy, then we have $\lambda_{d,d}(A) = 1$.*

The investigation of the structure of local cohomology modules $H^i_Y(\mathscr{F})$ was initiated by Grothendieck and is a very interesting subject in a field of commutative algebra, where Y is a closed subscheme of a scheme X and $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module. Although several authors have developed very interesting results and deep theories (see, for example, [**F1**], [**F2**], [**G1**], [**G2**], [**Ha1**], [**Ha3**], [**Ha4**], [**HaS**], [**HoR**], [**HuK**], [**HuL**], [**O**], [**PS**], [**Sh**]), these modules are still very mysterious. For finiteness property (for example, cofiniteness, finiteness of Bass numbers), several results are known (see, for example, [**D**], [**Yo**], [**DM**], [**Me**]. See also, [**Mo**], [**T**] and [**Ya**] for results of local cohomology modules with supports in monomial ideals). In particular, Huneke and Sharp [**HuS**] and Lyubeznik [**L1**] proved remarkable results and further, Lyubeznik defined a numerical invariant of local rings with respect to local cohomology modules [**L1**, theorem-definition 4·1]:

Definition 1. Let A be a local ring of dimension d which admits a surjective ring homomorphism $\pi: R \to A$, where R is a regular local ring of dimension n containing a field. Set $I = \ker \pi$ and let $\mathfrak{m}$ be the maximal ideal of R. Then the Bass number $\mu_p(\mathfrak{m}, H^{n-i}_I(R))$ is finite and depends only on A, i and p, but neither on R nor on π. We denote this invariant by $\lambda_{p,i}(A)$, and we call this number the Lyubeznik number (or the (p,i)-Lyubeznik number).

A complete local ring containing a field is always a surjective image of a regular local ring containing a field. So, if A is a local ring containing a field, but not necessarily a surjective image of a regular local ring containing a field, one can set $\lambda_{p,i}(A) = \lambda_{p,i}(A^\wedge)$, where $A^\wedge$ is the completion of A with respect to the maximal ideal.

We recall some basic properties of $\lambda_{p,i}$ (cf. [**L1**, (4·4 i–v)]).

THEOREM 2 (Lyubeznik). *Let A be a local ring of dimension d containing a field. Then the following assertions hold*:

(i) $\lambda_{p,i}(A) = 0$ *if* $i > d$;

(ii) $\lambda_{p,i}(A) = 0$ *if* $p > i$;

(iii) $\lambda_{d,d}(A) \neq 0$;

(iv) *if A is analytically normal, then* $\lambda_{d,d}(A) = 1$;

(v) *if A is a complete intersection, then* $\lambda_{d,d}(A) = 1$.

These results lead Lyubeznik to ask the following question [**L1**, question 4·5].

Question 1 (Lyubeznik). Is it true that $\lambda_{d,d}(A) = 1$ for all A?

Recently Walther answered this question negatively [**W**, proposition 3·2], using the Brodmann sequence, whose rings are not Cohen–Macaulay. So we refine the above question as follows:

Question 2. Is it true that $\lambda_{d,d}(A) = 1$ for Cohen–Macaulay rings A?

This is true for Cohen–Macaulay local rings A of characteristic p by the result of [**PS**, proposition 4·1], since the spectral sequence $H^p_{\mathfrak{m}} H^q_I(R) \Rightarrow H^{p+q}_{\mathfrak{m}}(R)$ degenerates. In this paper, we affirmatively answer Question 2, that is we will prove that if A is a Cohen–Macaulay local ring containing a field of dimension d, then the highest Lyubeznik number $\lambda_{d,d}(A)$ is one.

This paper is dedicated to Professor Yukitoshi Hinohara on the occasion of his seventieth birthday.

2. *Proof of the lemmas*

Definition 2. Let A be a ring. For a positive integer k, we say that A satisfies the Serre (S_k)-condition if it holds $\operatorname{depth}(A_P) \geqslant \inf(k, \operatorname{ht}(P))$ for all $P \in \operatorname{Spec}(A)$ (cf. [**Ma1**, (17. I), p. 125]).

Remark 1. If a catenary local ring A satisfies the Serre (S_2)-condition, then A is equidimensional, that is to say, all the irreducible components of $X = \operatorname{Spec} A$ have the same dimension (cf. [**Ha2**, remark 2·4·1, p. 503]).

The following lemma is probably well known to experts.

LEMMA 1. *Let $\phi: (A, \mathfrak{m}) \to (A', \mathfrak{m}')$ be a local homomorphism of local rings A, A' with the maximal ideals $\mathfrak{m}$, $\mathfrak{m}'$, respectively. Suppose that the going-down theorem holds for ϕ (see* [**Ma1**, (5. A), p. 31] *for the definition) and $\mathfrak{m}' = \mathfrak{m}A$. If the dimension of A is equal to d, then the dimension of A' is equal to d.*

Proof. First note that if I is an ideal of A, then the extension $\sqrt{I}A'$ of the radical of I is contained in the radical $\sqrt{IA'}$ of the extension of I. Let $x_1, \ldots, x_d$ be a system of parameters of A. Then the radical $\sqrt{(x_1, \ldots, x_d)}$ is equal to $\mathfrak{m}$. It follows that $\sqrt{(x_1, \ldots, x_d)A'} \supseteq \sqrt{(x_1, \ldots, x_d)}A' = \mathfrak{m}A' = \mathfrak{m}'$. Since $\mathfrak{m}'$ is the maximal ideal of

A', we have $\sqrt{(x_1, \ldots, x_d)A'} = \mathfrak{m}'$. Hence it holds that $\dim A' \leqslant d = \dim A$, since the arithmetic rank of $\mathfrak{m}'$ is not more than d. On the other hand, since $\mathfrak{m}$ is the maximal ideal, we have $\mathfrak{m}A' \cap A = \mathfrak{m}$. Since the going-down theorem holds for ϕ and $\mathfrak{m}A' \cap A = \mathfrak{m}$, we have the inequality $\operatorname{ht}\mathfrak{m} \leqslant \operatorname{ht}\mathfrak{m}'$ by the definition that the going-down theorem holds between A and A' (cf. [**Ma1**, (5. A), p. 31]). Then it follows that $\dim A \leqslant \dim A'$. We therefore conclude that $\dim A = \dim A' = d$.

We shall collect some results as lemmas for the proof of Proposition 1 and Theorem 1.

Lemma 2. *Let $(A, \mathfrak{m})$ be a local ring with the maximal ideal $\mathfrak{m}$. Then the following assertions hold:*

(i) *$(A^\wedge, \mathfrak{m}A^\wedge)$ is a local ring with the maximal ideal $\mathfrak{m}A^\wedge$;*

(ii) *if A is Cohen–Macaulay (respectively regular), then $A^\wedge$ is Cohen–Macaulay (respectively regular);*

(iii) *if A is a homomorphic image of a Cohen–Macaulay local ring and satisfies the Serre (S_k)-condition, then $A^\wedge$ satisfies the Serre (S_k)-condition for a positive integer k;*

(iv) *the natural map $A \to A^\wedge$ is a faithfully flat extension;*

(v) *if $A = R/I$ for some local ring R and an ideal I of R, then it holds that $A^\wedge = R^\wedge/IR^\wedge$;*

(vi) *if the dimension of A is equal to d, then the dimension of $A^\wedge$ is equal to d,*

where $A^\wedge$ is the completion of A with respect to the maximal ideal $\mathfrak{m}$.

Proof. Statement (i) follows from [**Ma2**, theorem, section 8 (4), p. 63] and statement (ii) follows from [**Ma2**, theorem 17·5 (ii), p. 136] and [**Ma2**, lines 13–14 in the proof of theorem 19·5 (ii), p. 157]. Further, statement (iii) follows from [**Ma2**, exercises to section 23, 23·2, p. 185], (iv) follows from [**Ma1**, (4. A), corollary, p. 27] and [**Ma1**, (23. L), corollary 1, p. 170], and (v) follows from [**Ma2**, theorem 8·11, p. 61]. Finally, since the natural map $A \to A^\wedge$ is flat by (iv), the going-down theorem holds between A and $A^\wedge$ by [**Ma1**, (5. D), theorem 4, p. 33]. Therefore assertion (vi) follows from Lemma 1.

Lemma 3. *Let $(A, \mathfrak{m})$ be a local ring with the maximal ideal $\mathfrak{m}$. Then the following assertions hold:*

(i) *A^{sh} is a local ring with the maximal ideal $\mathfrak{m}A^{sh}$;*

(ii) *if A is Cohen–Macaulay (respectively regular), then A^{sh} is Cohen–Macaulay (respectively regular);*

(iii) *if A satisfies the Serre (S_k)-condition, then A^{sh} satisfies the Serre (S_k)-condition for a positive integer k;*

(iv) *the natural map $A \to A^{sh}$ is a faithfully flat extension;*

(v) *if $A = R/I$ for some local ring R and an ideal I of R, then it holds that $A^{sh} = R^{sh}/IR^{sh}$;*

(vi) *if the dimension of A is equal to d, then the dimension of A^{sh} is equal to d,*

where A^{sh} is a strict henselization of A (see [**G3**, (18·8), p. 144] *for the definition).*

Proof. Statements (i) and (iv) follow from [**G3**, proposition (18·8·8) (i), p. 146] and [**G3**, proposition (18·8·8) (iii), p. 147]. Statements (ii) and (iii) follow from [**G3**, corollary (18·8·13) (a–c), p. 149], and (v) follows from [**Mil**, p. 38, line 27]. Finally, since the natural map $A \to A^{sh}$ is flat by (iv), the going-down theorem holds between A and A^{sh} by [**Ma1**, (5. D), theorem 4, p. 33]. Therefore assertion (vi) follows from (i) and Lemma 1.

LEMMA 4. *Let $(A, \mathfrak{m})$ be a local ring with the maximal ideal $\mathfrak{m}$. Let B be the ring $((A^\wedge)^{sh})^\wedge$. Then the following assertions hold:*

(i) *B is a local ring with the maximal ideal $\mathfrak{m}B$;*

(ii) *if A is Cohen–Macaulay (respectively regular), then B is Cohen–Macaulay (respectively regular);*

(iii) *if A is a homomorphic image of a Cohen–Macaulay local ring and satisfies the Serre (S_k)-condition, then B satisfies the Serre (S_k)-condition for a positive integer k;*

(iv) *the natural map $A \to B$ is a faithfully flat extension;*

(v) *if $A = R/I$ for some local ring R and an ideal I of R, then it holds that $B = ((R^\wedge)^{sh})^\wedge/I((R^\wedge)^{sh})^\wedge$;*

(vi) *if the dimension of A is equal to d, then the dimension of B is equal to d,*

where $A^\wedge$ is the completion of A with respect to the maximal ideal, and A^{sh} is a strict henselization of A.

Proof. We only prove statement (iii). Repeating a similar argument, the other assertions follow immediately from Lemmas 1–3, and [**Ma1**, (4. B), p. 27].

Let R be a Cohen–Macaulay local ring which surjectively maps to a local ring A. We denote the kernel of the surjection by I. Then A is isomorphic to R/I. If A satisfies the Serre (S_k)-condition, then $A^\wedge$ satisfies the Serre (S_k)-condition by (iii) of Lemma 2, since A is a homomorphic image of a Cohen–Macaulay local ring R. Since R is a Cohen–Macaulay local ring, $R^\wedge$ is also a Cohen–Macaulay local ring by (ii) of Lemma 2. Further $A^\wedge$ is isomorphic to $(R/I)^\wedge = R^\wedge/IR^\wedge$ by (v) of Lemma 2. Hence $A^\wedge$ is a homomorphic image of a Cohen–Macaulay local ring $R^\wedge$.

Since $A^\wedge$ satisfies the Serre (S_k)-condition, $(A^\wedge)^{sh}$ satisfies the Serre (S_k)-condition by (iii) of Lemma 3. Since $R^\wedge$ is a Cohen–Macaulay local ring, $(R^\wedge)^{sh}$ is also a Cohen–Macaulay local ring by (ii) of Lemma 3. Further $(A^\wedge)^{sh}$ is isomorphic to $((R/I)^\wedge)^{sh} = (R^\wedge)^{sh}/I(R^\wedge)^{sh}$ by (v) of Lemma 3. Hence $(A^\wedge)^{sh}$ is a homomorphic image of a Cohen–Macaulay local ring $(R^\wedge)^{sh}$.

Now since $(A^\wedge)^{sh}$ is a homomorphic image of a Cohen–Macaulay local ring $(R^\wedge)^{sh}$ and $(A^\wedge)^{sh}$ satisfies the Serre (S_k)-condition, $B = ((A^\wedge)^{sh})^\wedge$ satisfies the Serre (S_k)-condition by (iii) of Lemma 2. The proof of assertion (iii) is completed.

LEMMA 5. *Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension n containing a field, and I an ideal of R of dimension $d > 1$. If R/I satisfies the Serre (S_2)-condition, then the following assertions hold:*

(i) $\operatorname{inj.dim}_R H_I^{n-d}(R) = d$;

(ii) $\operatorname{inj.dim}_R H_I^j(R) < n - 1 - j$ *if* $j > n - d$,

where $\operatorname{inj.dim}_R T$ *is the injective dimension of an R-module T.*

Proof. Statement (i) is straightforward from [**L1**, (4·4iii)], so we only have to prove the assertion (ii). We proceed in several steps.

Step 1. First we prove that the 'Second Vanishing Theorem' holds for the local cohomology module $H_I^j(R)$, that is $H_I^j(R) = 0$ for $j \geqslant \dim R - 1$ (cf. [**Sp**, p. 143, line 15]).

Since $A = R/I$ is a homomorphic image of regular local ring R, $B = ((A^\wedge)^{sh})^\wedge = ((R^\wedge)^{sh})^\wedge/I((R^\wedge)^{sh})^\wedge$ is also a homomorphic image of regular local ring $((R^\wedge)^{sh})^\wedge$ by (ii) of Lemma 4. If $A = R/I$ satisfies the Serre (S_2)-condition, then the local ring $B = ((A^\wedge)^{sh})^\wedge = ((R^\wedge)^{sh})^\wedge/I((R^\wedge)^{sh})^\wedge$ also satisfies the Serre (S_2)-condition by (iii) of Lemma 4, and the dimension of B and $((R^\wedge)^{sh})^\wedge$ are equal to d and n respectively by (vi) of Lemma 4. It follows from [**Ha2**, corollary 2·4, p. 503] that the punctured spectrum of Spec (B) is connected, that is Spec A is formally geometrically connected (see [**HuL**, theorem 2·9, p. 79] for the definition). Hence since the dimension of R/I is greater than one, we have $H_I^n(R) = H_I^{n-1}(R) = 0$ by [**HuL**, theorem 2·9, p. 79].

Step 2. Next we shall prove that $H_I^j(R)_Q = 0$ for all prime ideals Q containing I with $\dim R/Q \geqslant n-1-j$ and $\dim R/Q < d-1$.

Indeed, let Q be any prime ideal of R containing I with $\dim R/Q \geqslant n-1-j$ and $\dim R/Q < d-1$.

(*a*) If $\dim R/Q \geqslant n-1-j$, then we have:

$$\begin{aligned} j &\geqslant n-1-\dim R/Q \\ &= \dim R - 1 - \dim R/Q \\ &= (\dim R - \dim R/Q) - 1 \\ &= \mathrm{ht}Q - 1 \\ &= \dim R_Q - 1, \end{aligned}$$

by [**Ma2**, section 31, lemma 2, p. 250].

(*b*) Further since R/I satisfies the Serre (S_2)-condition, R/I is equidimensional by Remark 1. Hence it holds that

$$\begin{aligned} \dim (R/I)_Q &= \mathrm{ht}Q/I \\ &= \dim R/I - \dim ((R/I)\big/(Q/I)), \end{aligned}$$

by [**Ma2**, section 31, lemma 2, p. 250] again. If $\dim R/Q < d-1$, then we have:

$$\begin{aligned} \dim R_Q/IR_Q &= \dim (R/I)_Q \\ &= \mathrm{ht}Q/I \\ &= \dim R/I - \dim ((R/I)/(Q/I)) \\ &= \dim R/I - \dim R/Q \\ &> d-(d-1) \\ &= 1, \end{aligned}$$

by [**Ma2**, section 31, lemma 2, p. 250].

We note that the localization $(R/I)_Q = R_Q/IR_Q$ satisfies the Serre (S_2)-condition and R_Q is also a regular local ring.

Now we consider the category of modules over the regular local ring R_Q. It then follows from Step 1 that the Second Vanishing Theorem holds for the local cohomology module $H^j_{IR_Q}(R_Q)$, that is, $H_I^j(R)_Q = 0$ for $j \geqslant \dim R_Q - 1$ and $\dim R_Q/IR_Q > 1$ by (*a*) and (*b*). We obtain the assertion of this step.

Step 3. In this step, we shall prove that $\dim \mathrm{Supp} H_I^j(R)_Q < n-1-j$ under the assumption that $j > n-d$.

Indeed, let Q be in $\mathrm{Supp} H_I^j(R)$, then we have $H_I^j(R)_Q = H^j_{IR_Q}(R_Q) \neq 0$. From the assumption $j > n - d$, we have

$$\begin{aligned} \dim R/Q &\leqslant n - 1 - j \\ &< n - 1 - (n - d) \\ &= d - 1. \end{aligned}$$

It follows from Step 2(*b*) that $\dim R_Q/IR_Q > 1$. Then it holds that

$$\begin{aligned} j &\leqslant \dim R_Q - 1 \\ &= \dim R - \dim R/Q - 1 \\ &= n - 1 - \dim R/Q, \end{aligned}$$

by the local Lichtenbaum–Hartshorne Vanishing Theorem and the Grothendieck Vanishing Theorem for the local cohomology module $H_I^j(R)_Q = H_{IR_Q}^j(R_Q) \neq 0$. It therefore follows from Step 2 that $\dim R/Q < n - 1 - j$ for all prime ideals Q in $\mathrm{Supp} H_I^j(R)$. We then conclude that $\dim \mathrm{Supp} H_I^j(R) < n - j - 1$.

Step 4. Since the inequality $\mathrm{inj.dim} H_I^j(R) \leqslant \dim \mathrm{Supp} H_I^j(R)$ holds by [**HuS**] and [**L1**, corollary 3·6, p. 52], the proof of the assertion (ii) is completed.

3. *Proof of the main results*

PROPOSITION 1. *Let R be a regular local ring containing a field of dimension n, I an ideal of R of dimension d > 1. If R/I satisfies the Serre (S_2)-condition, then we have* $\lambda_{d,d}(R/I) = 1$.

Proof. The functor $\Gamma_I(-)$ takes injectives into $\Gamma_{\mathfrak{m}}(-)$-acyclic objects in the category of R-modules. So by Grothendieck's spectral sequence, we obtain the following spectral sequence (cf. [**G1**, theorem A, p. 5]):

$$E_2^{p,q} = H_{\mathfrak{m}}^p H_I^q(R) \Longrightarrow H^{p+q} = H_{\mathfrak{m}}^{p+q}(R).$$

The spectral sequence has the differentials as follows:

$$E_r^{d-r,n-d-(1-r)} \longrightarrow E_r^{d,n-d} \longrightarrow E_r^{d+r,n-d+(1-r)}.$$

We shall prove that all the differentials that come into and go out of $E_r^{d,n-d}$ are 0 for all $r \geqslant 2$.

Now it follows from [**L1**, (4·4i), p. 54] that $H_{\mathfrak{m}}^p H_I^{n-i}(R) = 0$ for all $i > d$ and all $p \geqslant 0$. On the other hand, we calculate as follows:

$$\begin{aligned} -(-d + (1 - r)) &= d - 1 + r \\ &\geqslant d - 1 + 2 \\ &= d + 1 \\ &> d, \end{aligned}$$

for $r \geqslant 2$. Hence we have $E_2^{d+r,n-d+(1-r)} = H_{\mathfrak{m}}^{d+r} H_I^{n-d+(1-r)}(R) = 0$. So it holds that $E_r^{d+r,n-d+(1-r)} = 0$ for all $r \geqslant 2$. Thus it follows that the differentials $E_r^{d,n-d} \to E_r^{d+r,n-d+(1-r)}$ are 0 for all $r \geqslant 2$.

From Lemma 5, we calculate as follows:

$$\begin{aligned} \mathrm{inj.dim}\, H_I^{n-d-(1-r)}(R) &< n - 1 - (n - d - (1 - r)) \\ &= d - r. \end{aligned}$$

Hence we have $E_2^{d-r,n-d-(1-r)} = H_{\mathfrak{m}}^{d-r} H_I^{n-d-(1-r)}(R) = 0$. So it holds that

$E_r^{d-r,n-d-(1-r)} = 0$ for all $r \geqslant 2$. Thus it follows that the differentials $E_r^{d-r,n-d-(1-r)} \to E_r^{d,n-d}$ are 0 for all $r \geqslant 2$.

Further we have $E_2^{p,q} = 0$ for all $p \geqslant 0$, $q \geqslant 0$ with $p+q=n$, $p<d$ by Lemma 5. Since $\text{inj.dim} H_I^q(R) \leqslant \dim \text{Supp} H_I^q(R) \leqslant \dim V(I) = d$ by [**HuS**] and [**L1**, corollary 3·6, p. 52], it holds that $E_2^{p,q} = 0$ for all $p \geqslant 0$, $q \geqslant 0$ with $p+q=n$, $p>d$. It then follows that E_2-terms are all 0 in the total degree n except $E_2^{d,n-d}$.

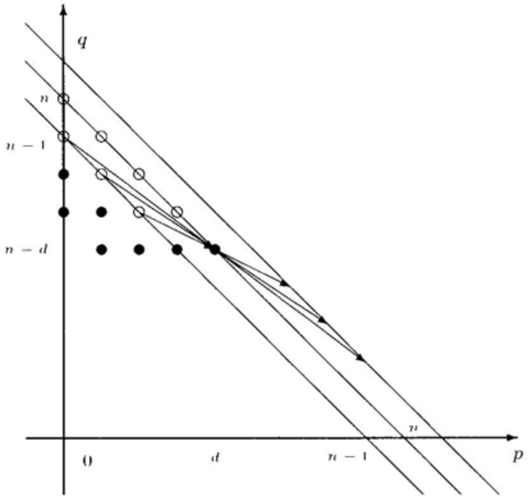

Fig. 1

We express E_2-terms in the above diagram (Fig. 1). The circles mean the vanishing of E_2-terms by Lemma 5. Furthermore all E_2-terms are 0 except the black circles.

Therefore the above spectral sequence collapses at $E_2^{d,n-d}$ and we have isomorphisms:

$$\begin{aligned} H_{\mathfrak{m}}^d H_I^{n-d}(R) &= E_2^{d,n-d} \\ &\simeq E_\infty^{d,n-d} \\ &\simeq H^n \\ &= H_{\mathfrak{m}}^n(R). \end{aligned}$$

Since R is a regular local ring, $H_{\mathfrak{m}}^n(R)$ is isomorphic to $E(k)$, where $E(k)$ is the injective hull of k. Since $H_{\mathfrak{m}}^d H_I^{n-d}(R)$ is isomorphic to $E(k)$, it therefore follows from [**L1**, lemma 1·4, p. 44] that $\lambda_{d,d}(A) = 1$. The proof of the proposition is completed.

Proof of Theorem 1. Completing the local ring A with respect to the topology defined by the maximal ideal, there is a surjection $R \to A^\wedge$ from a regular local ring R containing a fields to $A^\wedge$ by Cohen's structure theorem. We denote its kernel by I and the maximal ideal of R by $\mathfrak{m}$. $A^\wedge = R/I$ is also Cohen-Macaulay by (i) of Lemma 3. Especially $A^\wedge = R/I$ satisfies the Serre (S_2)-condition by [**Ma1**, (17. I), p. 125, lines 8–9]. Therefore the theorem follows from Proposition 1.

COROLLARY 1. *Let A be a local ring containing a field of dimension $d > 1$, complete with the topology defined by the maximal ideal. If A satisfies the Serre (S_2)-condition, then we have $\lambda_{d,d}(A) = 1$.*

Proof. By Cohen's structure theorem, there is a surjection from a regular local ring R to A. Therefore the assertion follows from Proposition 1.

Remark 2. If the dimension of a ring A is equal to one, then it always holds that $\lambda_{1,1}(A) = 1$.

Example 1. The converse of the theorem, the proposition and the corollary do not hold in general. The following example is essentially due to Kazufumi Eto. Let R be the localization of $k[x_1, x_2, x_3, x_4, x_5, x_6]$ by $(x_1, x_2, x_3, x_4, x_5, x_6)$, $I = (x_1, x_2, x_3) \cap (x_2, x_3, x_4) \cap (x_3, x_4, x_5) \cap (x_4, x_5, x_6) \cap (x_5, x_6, x_1)$ and $P = (x_1, x_2, x_3, x_5, x_6)$. Here we note that I is equidimensional and $\dim R/I = 3$. Then we have $\dim(R/I)_P = 2$ and $IR_P = (x_1, x_2, x_3)R_P \cap (x_5, x_6, x_1)R_P$. It follows that the punctured spectrum of $\mathrm{Spec}(R/I)_P$ is disconnected, that is $\mathrm{Spec}\, R/I$ is not locally connected in codimension one (see [**Ha2**, definition, p. 500] for the definition). It follows from [**Ha2**, corollary 2·4, p. 503] that $\mathrm{Spec}\, R/I$ does not satisfy the (S_2)-condition and therefore R/I is not Cohen–Macaulay.

On the other hand, Yanagawa shows that for pure square-free monomial ideals, the highest Lyubeznik number is one if and only if the corresponding ring is connected in codimension one (cf. [**Ya**, corollary 3·16]). The local space $\mathrm{Spec}\, R/I$ is connected in codimension one and I is a pure square-free monomial ideal in R. It follows from [**Ya**, corollary 3·16] that $\lambda_{3,3}(R/I) = 1$. While he used the combinatorial argument to prove [**Ya**, corollary 3·16], one can find the direct proof of the result that $\lambda_{3,3}(R/I) = 1$ in [**EK**], by which we could point out that the earlier version of [**Ya**, corollary 3·16] was incorrect.

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REFERENCES

[**AM**] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Series in Math. (Reading, Massachusetts, 1969).

[**D**] D. Delfino. On cofiniteness of local cohomology modules. *Math. Proc. Camb. Phil. Soc.* **115** (1994), 421–429.

[**DM**] D. Delfino and T. Marley. Cofinite modules and local cohomology. *J. Pure Appl. Algebra* **121** (1997) 45–52.

[**EK**] K. Eto and K.-I. Kawasaki. In preparation.

[**F1**] G. Faltings. Über die Annulatoren lokaler Kohomologiegruppen. *Archiv der Math.* **30** (1978), 473–476.

[**F2**] G. Faltings. Über lokale Kohomologiegruppen hoher Ordnung. *J. Reine Angew. Math.* **313** (1980) 43–51.

[**G1**] A. Grothendieck. *Local cohomology, notes by Hartshorne* Lect. Notes Math. vol. **41** (Springer-Verlag, 1967).

[**G2**] A. Grothendieck. *Cohomologie locale des faisceaux cohérents et Théorèmes de Lefschetz locaux et globaux* (*SGA2*) (North-Holland, 1968).

[**G3**] A. Grothendieck. Éléments de Géométrie Algébrique IV. *Publ. Math. I.H.E.S.* **32** (1967).

[**Ha1**] R. Hartshorne. Affine duality and cofiniteness. *Invent. Math.* **9** (1970), 145–164.

[**Ha2**] R. Hartshorne. Complete intersections and connectedness. *Amer. J. Math.* **84** (1962), 497–508.

[**Ha3**] R. Hartshorne. Cohomological dimension of algebraic varieties. *Ann. Math.* **88** (1968), 403–450.

[**Ha4**] R. Hartshorne. *Lectures on the Grothendieck Duality theory*. Lect. Notes Math. vol. **20** (Springer-Verlag, 1966).

[**HoR**] M. Hochster and J. Roberts. The purity of the Frobenius and Local cohomology. *Adv. Math.* **21** (2) (1976), 117–172

[HaO] R. Hartshorne and A. Ogus. On the factoriality of local rings of small embedding codimension. *Comm. in Alg.* **1** (5) (1974), 415–437.

[HaS] R. Hartshorne and R. Speiser. Local cohomological dimension in characteristic p. *Ann. Math.* **105** (1977), 45–79.

[HuL] C. Huneke and G. Lyubeznik. On the vanishing of local cohomology modules. *Invent Math.* **102** (1990), 73–93.

[HuK] C. Huneke and J. Koh. Cofiniteness and vanishing of local cohomology modules. *Math Proc. Camb. Phil. Soc.* **110** (1991), 421–429.

[HuS] C. Huneke and R. Y. Sharp. Bass numbers of local cohomology modules. *Trans. Amer. Math. Soc.* **339** (1993), 765–779.

[L1] G. Lyubeznik. Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra). *Invent. Math.* **102** (1993), 41–55.

[L2] G. Lyubeznik. F-modules: applications to local cohomology and D-modules in characteristic $p > 0$. *J. Reine Angew. Math.* **491** (1997), 65–130.

[Ma1] H. Matsumura. *Commutative algebra* (Benjamin/Cummings, 2nd ed., 1980).

[Ma2] H. Matsumura. *Commutative ring theory*. Cambridge Studies in Advance Mathematics, **8** (Cambridge University Press, 1986).

[Me] L. Melkersson. Properties of cofinite modules and applications to local cohomology. *Math. Proc. Camb. Phil. Soc.* **110** (1999), 421–429.

[Mil] J. S. Milne. *Étale Cohomology* (Princeton University Press, 1986).

[Mo] J. A. Montaner. Characteristic cycles of local cohomology modules of monomial ideals. preprint.

[O] A. Ogus. Local cohomological dimension of algebraic varieties. *Ann. Math.* **98** (1973), 1–34.

[PS] C. Peskine and L. Szpiro. Dimension projective finie et cohomologie locale. *Publ. Math. I.H.E.S.* **42** (1973), 323–395.

[Sh] R. Y. Sharp. The Frobenius homomorphism and local cohomology in regular local ring of positive characteristic. *J. Pure Appl. Algebra* **71** (2–3) (1970), 313–317.

[Sp] R. Speiser. Cohomological dimension of non-complete hypersurfaces. *Invent. Math.* **21** (1973), 143–150.

[T] N. Terai. Local cohomology modules with respect to monomial ideals. preprint.

[W] U. Walther. Some calculations on the Lyubeznik numbers of a local ring. preprint.

[Yo] K.-i. Yoshida. Cofiniteness of local cohomology modules for dimension one ideals. *Nagoya J. Math.* **147** (1997), 179–191.

[Ya] K. Yanagawa. Bass numbers of local cohomology modules with supports in monomial ideal. *Math. Proc. Camb. Phil. Soc.* **131** (2001), 45–60.