# Entropy of Probability Measures on Finite Commutative Hypergroups

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**Abstract**

The purpose of this paper is to investigate entropy of probability measures on finite commutative hypergroups. In fact, we give a notion of entropy which is compatible with entropy of random walks on finite symmetric regular graphs. We study some fundamental propaties of the entropy concerning with maximality. (AMS Subject Classification : 43A62, 20N20.)

**Key Words** : hypergroup, entropy, random walk

## 1. Introduction

Roughly speaking, the hypergroup convolution is a probabilistic extension of the group convolution. The concept of convolution of measures on a locally compact group has been generalized beyond the group case in the axiomatic setting of a hypergroup, due to C.F. Dunkl, R.I. Jewett, and R. Spector around 1975.

In this paper we establish a notion of entropy of probability measures on finite commutative hypergroups which is compatible with usual entropy of random walks on finite symmetric regular graphs. Wildberger[10] studied a certain entropy of probability measure on finite hypergroups related with information theory. Our definition of entropy is different from his notion of entropy.

Let $K = \{c_0, c_1, \cdots, c_n\}$ be a finite commutative hypergroup with the $*$-algebra $A(K)$. We call the invariant measure $\mu = \mu_K = \sum_{k=0}^{n} w(c_k)c_k$ on $K$ the *canonical* Haar measure of $K$.

For a probability measure $\nu = \sum_{k=0}^{n} a_k c_k$ on $K$, we define a entropy $S_\mu(\nu)$ of $\nu$ relative to $\mu$ by

$$S_\mu(\nu) = -\nu\left(\log\frac{d\nu}{d\mu}\right) = -\sum_{k=0}^{n} a_k \log\frac{a_k}{w(c_k)}.$$

Let $\nu_0$ denote the normalized Haar measure of $K$ which is given by $\nu_0 = \dfrac{1}{w(K)}\mu$.

Then we have the following results.

In Theorem 1 we show $0 \leq S_\mu(\nu) \leq \log w(K)$ and we characterize the probability measure $\nu$ such that the entropy $S_\mu(\nu)$ attains the maximum value.

In Theorem 2 we show the following. Let $H = (H, A(H))$ be a generalized orbital hypergrgroup of $K = (K, A(K))$ by the conditional expectation $E$ from $A(K)$ onto $A(H)$ such that $H = E(K)$. Then $\mu_H = E(\mu_K)$ holds for the canonical Haar measures $\mu_K$ of $K$ and $\mu_H$ of $H$. For a probability measure $\nu$ on $K$ we have $S_{\mu_K}(\nu) \leq S_{\mu_H}(E(\nu))$. Moreover, the equality $S_{\mu_K}(\nu) = S_{\mu_H}(E(\nu))$ holds if and only if $\nu = E(\nu) \in A(H)$.

This work has been done by developing some results in bachelor's thesis[2] by the first author in 2007.

## 2. Preliminaries

We recall some notions and facts on finite commutative hypergroups from Bloom-Heyer's Book[1] and Wildberger's report[9]. $K := (K, A)$ is called a *finite commutative hypergroup* if the following conditions (1)～(6) are satisfied.

(1) $A$ is a $*$-algebra over $\mathbb{C}$ with the unit $c_0$.
(2) $K = \{c_0, c_1, \cdots, c_n\}$ is a linear basis of $A$.
(3) $K^* = K$.

(4) $c_i c_j = \sum_{k=0}^{n} n_{ij}^k c_k$, where $n_{ij}^k$ is a non-negative real number such that
$c_i^* = c_j \Longleftrightarrow n_{ij}^0 > 0$,
$c_i^* \neq c_j \Longleftrightarrow n_{ij}^0 = 0$.

(5) $\sum_{k=0}^{n} n_{ij}^k = 1$ for any $i, j$.

(6) $c_i c_j = c_j c_i$ for any $i, j$.

We often denote $A$ by $A(K)$ for $K = (K, A)$. The *weight* of an element $c_i \in K$ is defined by $w(c_i) := (n_{ij}^0)^{-1}$ where $c_j = c_i^*$, and the *total weight* of $K$ is given by $w(K) := \sum_{i=0}^{n} w(c_i)$.

Let $M^1(K)$ denote the set of probability measures on $K$, i.e.

$$M^1(K) := \{\nu = \sum_{k=0}^{n} a_k c_k : a_k \geq 0 \ (k = 0, 1, \cdots, n), \sum_{k=0}^{n} a_k = 1\}.$$

For $\nu = \sum_{k=0}^{n} a_k c_k \in A(K)$, *support* of $\nu$ is defined by

$$\mathrm{supp}(\nu) := \{c_k : a_k \neq 0, \ k = 0, 1, \cdots, n\}.$$

Let $\omega(K)$ denote the *normalized Haar measure* of $K$ which is given by

$$\omega(K) = \sum_{k=0}^{n} \frac{w(c_k)}{w(K)} c_k.$$

Let $A$ be a $*$-algebra with the unit $c_0$ and $B$ be a $*$-subalgebra of $A$ with the unit $c_0$. Then a linear mapping $E$ from $A$ onto $B$ is called a conditional expectation if the following conditions are satisfied.

(1) $E(c_0) = c_0$.

(2) $E(yxz) = yE(x)z$ for $x \in A$, $y, z \in B$.

(3) $E(x^* x) \geq 0$.

Let $H = (H, A(H))$ and $K = (K, A(K))$ be finite hypergroups such that the $*$-algebra $A(H)$ is realized in the $*$-algebra $A(K)$. We call $H$ a *generalized orbital hypergroup* of $K$ if there exists a conditional expectation $E$ from $A(K)$ onto $A(H)$ such that $H = E(K)$. This notion is a generalization of a usual orbital hypergroup.

## 3. Entropy of probability measures

Let $K = \{c_0, c_1, \cdots, c_n\}$ be a finite commutative hypergroup with the $*$-algebra $A(K)$. We call the invariant measure $\mu_K = \sum_{k=0}^{n} w(c_k) c_k$ on $K$ the *canonical* Haar measure of $K$. This $\mu_K$ is often denoted by $\mu$ when $K$ is obvious. For a probability measure $\nu = \sum_{k=0}^{n} a_k c_k$ on $K$, we define a entropy $S_\mu(\nu)$ of $\nu$ relative to $\mu$ by

$$S_\mu(\nu) = -\nu\left(\log \frac{d\nu}{d\mu}\right) = -\sum_{k=0}^{n} a_k \log \frac{a_k}{w(c_k)}.$$

Let $\nu_0$ denote the normalized Haar measure of $K$ which is given by $\nu_0 = \frac{1}{w(K)} \mu$. Then we have the following theorem.

**Theorem 1.** The entoropy $S_\mu(\nu)$ is non-negative and $S_\mu(\nu) \leq \log w(K)$. The entropy $S_\mu(\nu)$ attains the maximam value $\log w(K)$ if and only if $\nu = \nu_0$. Moreover, $S_\mu(\nu) = 0$ if and only if $a_k = 1$ for some $k$ such that $w(c_k) = 1$.

**Proof.** By the fact that $0 \leq \frac{a_k}{w(c_k)} \leq 1$, $-a_k \log \frac{a_k}{w(c_k)} \geq 0$. Then it is clear that $S_\mu(\nu) \geq 0$. Suppose that $S_\mu(\nu) = 0$. Then $-a_k \log \frac{a_k}{w(c_k)} = 0$ for all $k$. This implies that $\frac{a_k}{w(c_k)} = 0$ or 1. If $\frac{a_k}{w(c_k)} = 1$ for some $k$ then $a_k = w(c_k)$. Since $0 \leq a_k \leq 1$ and $w(c_k) \geq 1$, we obtain $a_k = 1$ and $w(c_k) = 1$. We note that $a_j = 0$ for all $j$ such that $j \neq k$. Moreover, applying Jensen's inequality, it is easy to see that $S_\mu(\nu) = \log w(K)$ if and only if $\frac{w(c_k)}{a_k} = w(K)$ for all $k$, namely $a_k = \frac{w(c_k)}{w(K)}$. This implies that $\nu = \nu_0$.

[Q.E.D.]

## 4. Entropy and generalized orbital hypergroups

Let $H = (H, A(H))$ and $K = (K, A(K))$ be finite commutative hypergroups such that the $*$-algebra $A(H)$ is realized in the $*$-algebra $A(K)$. We call $H$ a generalized orbital hypergroup of $K$ if there exists a conditional expectation $E$ from $A(K)$ onto $A(H)$ such that $H = E(K)$. When an action $\alpha$ of a finite group $G$ on a hypergroup $K$ is given, an orbital hyeprgroup $H = K^\alpha$ is defined by the conditional expectation $E$ by

$$E(x) = \frac{1}{|G|} \sum_{g \in G} \alpha_g(x) \ \text{ for } x \in A(K).$$

We note that many hypergroups are obtained as generalized orbital hypergroups which are not necessarily usual orbital hypergroups. Refer to our paper(4).

**Theorem 2.** Let $H = (H, A(H))$ be a generalized orbital hypergrgroup of $K = (K, A(K))$ by a conditional expectation $E$ from $A(K)$ onto $A(H)$ such that $H = E(K)$. Then $\mu_H = E(\mu_K)$ holds for the canonical Haar measures $\mu_K$ of $K$ and $\mu_H$ of $H$. For a probability measure $\nu$ on $K$ we have $S_{\mu_K}(\nu) \leq S_{\mu_H}(E(\nu))$. Moreover, the equality $S_{\mu_K}(\nu) = S_{\mu_H}(E(\nu))$ holds if and only if $\nu = E(\nu) \in A(H)$.

**Proof.** Let $K$ and $H$ be given by $K = \{c_0, c_1, \cdots, c_n\}$ and $H = \{d_0, d_1, \cdots, d_m\}$, where $c_0$ is the unit of $K$ and $d_0$ is the unit of $H$, and $c_0 = d_0$. For each $d_j \in H$, set

$$K(j) = \{c \in K : E(c) = d_j\}$$
$$= \{c_1(j), c_2(j), \cdots, c_{n_j}(j)\}$$

We note that

$$K = \bigcup_{j=0}^{m} K(j) \text{ and } \sum_{j=0}^{m} n_j = n$$

Moreover, it is easy to see that each $d_j \in H$ is written as

$$d_j = \sum_{i=1}^{n_j} a_i(j) c_i(j) \text{ where } \sum_{i=1}^{n_j} a_i(j) = 1.$$

By this fact, we see that $d_j \mu_K = \mu_K$ for each $d_j \in H$. Hence $d_j E(\mu_K) = E(d_j \mu_K) = E(\mu_K)$. This implies that the measure $E(\mu_K)$ is $H$-invariant so that $E(\mu_K)$ is a Haar measure of $H$. Therefore $E(\mu_K)$ is written by $E(\mu_K) = c\mu_H$ for some constant $c > 0$. Since $\mu_K$ and $\mu_H$ is represented as

$$\mu_K = \sum_{k=0}^{n} w(c_k) c_k, \ \mu_H = \sum_{j=0}^{m} w(d_j) d_j,$$

and $E(c_0) = d_0$, we see that the constant $c$ must be 1 so that $\mu_H = E(\mu_K)$ holds. The canonical Haar measure $\mu_K$ of $K$ is given by

$$\mu_K = \sum_{j=0}^{m} \sum_{i=1}^{n_j} w(c_i(j)) c_i(j),$$

where

$$K(j) = \{c_1(j), c_2(j), \cdots, c_{n_j(j)}\} \text{ and } K = \bigcup_{j=0}^{m} K(j).$$

Since $E(c_i(j)) = d_j$,

$$E(\mu_K) = \sum_{j=0}^{m} \left( \sum_{i=1}^{n_j} w(c_i(j)) \right) d_j.$$

By the fact that $\mu_H = E(\mu_K)$, we see that

$$w(d_j) = \sum_{i=1}^{n_j} w(c_i(j)).$$

For a probability measure $\nu = \sum_{k=0}^{n} a_k c_k = \sum_{j=0}^{m} \sum_{i=1}^{n_j} a_i(j) c_i(j)$ of $K$, $E(\nu)$ is given by

$$E(\nu) = \sum_{j=0}^{m} \left( \sum_{i=1}^{n_j} a_i(j) \right) d_j = \sum_{j=0}^{m} b_j d_j,$$

where $b_j = \sum_{i=1}^{n_j} a_i(j)$. Then we get the following equalities.

$$S_{\mu_K}(\nu) = -\sum_{j=0}^{m} \sum_{i=1}^{n_j} a_i(j) \log \frac{a_i(j)}{w(c_i(j))},$$

$$S_{\mu_H}(E(\nu)) = -\sum_{j=0}^{m} b_j \log \frac{b_j}{w(d_j)}.$$

We may assume that $a_i(j) > 0$. Hence we see that

$$-\sum_{i=1}^{n_j} \frac{a_i(j)}{b_j} \log \frac{a_i(j)}{w(c_i(j))} = \sum_{i=1}^{n_j} \frac{a_i(j)}{b_j} \log \frac{w(c_i(j))}{a_i(j)}$$

$$\leq \log \sum_{i=1}^{n_j} \frac{a_i(j)}{b_j} \frac{w(c_i(j))}{a_i(j)} = \log \sum_{i=1}^{n_j} \frac{w(c_i(j))}{b_j} = -\log \frac{b_j}{w(d_j)},$$

by Jensens' inequality. Hence we see that

$$-\sum_{i=1}^{n_j} a_i(j) \log \frac{a_i(j)}{w(c_i(j))} \leq -b_j \log \frac{b_j}{w(d_j)}.$$

Therefore we obtain that $S_{\mu_K}(\nu) \leq S_{\mu_H}(E(\nu))$. Moreover, it is also obtained that the equality holds if and only if $\frac{w(c_i(j))}{a_i(j)} = \frac{w(d_j)}{b_j}$ for all $i = 1, 2, \cdots, n_j$. This implies that

$$\sum_{i=1}^{n_j} a_i(j) c_i(j) = b_j d_j, \text{ namely, } \nu = E(\nu) \in A(H).$$

[Q.E.D.]

**Remark.** When $H$ is an orbital hypergroup of $K$ by an action $\alpha$ of a group $G$ on $K$, the condition $\nu = E(\nu) \in A(H)$ is equivalent to say that $\nu$ is $\alpha$-invariant.

Therefore we note that the equality $S_{\mu_K}(\nu) = S_{\mu_H}(E(\nu))$ holds if and only if $\nu$ is $\alpha$-invariant.

**Example.** Let $K = \{c_0, c_1, c_2\}$ be the cyclic group $\mathbb{Z}_3$ of order three such that $c_1^3 = c_0$, $c_1^2 = c_2$, $c_1^* = c_2$, and $c_2^* = c_1$. Let $H = \{d_0, d_1\}$ be the hypergroup of order two arising from random walk on edges of a regular triangle where $d_1^2 = \frac{1}{2}d_0 + \frac{1}{2}d_1$, $d_1^* = d_1$, and $w(d_1) = 2$. Then we note that the hypergroup $H$ is realized in $A(K)$ by the relation $d_0 = c_0$ and $d_1 = \frac{1}{2}c_1 + \frac{1}{2}c_2$. We can interpret that this hypergroup $H$ is an orbital hypergroup by an action $\alpha$ of the group $G = \{e, g\}$ $(g^2 = e)$ of order two on $K$ such that $\alpha_g(c_1) = c_2$ and $\alpha_g(c_2) = c_1$. We can also interpret that this hypergroup $H$ is a generalized orbital hypergroup by the conditional expectation $E$ from $A(K)$ onto $A(H)$ such that $E(c_0) = d_0$ and $E(c_1) = E(c_2) = \frac{1}{2}c_1 + \frac{1}{2}c_2 = d_1$. In this case the Haar measures $\mu_K$ of $K$ and $\mu_H$ of $H$ are given by

$$\mu_K = c_0 + c_1 + c_2,$$
$$\mu_H = d_0 + 2d_1.$$

We note that

$$E(\mu_K) = E(c_0) + E(c_1) + E(c_2)$$
$$= d_0 + d_1 + d_1$$
$$= d_0 + 2d_1$$
$$= \mu_H.$$

For $\nu = a_0c_0 + a_1c_1 + a_2c_2 \in M^1(K)$, $E(\nu) = a_0d_0 + (a_1 + a_2)d_1$, we have

$$S_{\mu_K}(\nu) = -a_0 \log a_0 - a_1 \log a_1 - a_2 \log a_2,$$
$$S_{\mu_H}(E(\nu)) = -a_0 \log a_0 - (a_1 + a_2) \log \frac{a_1 + a_2}{2}.$$

Theorem asserts that the equality $S_{\mu_K}(\nu) = S_{\mu_H}(E(\nu))$ holds if and only if $a_1 = a_2$ which is equivalent to say that $\nu$ is an $\alpha$-invariant measure.

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