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On finiteness of the Betti numbers of local cohomology module

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Abstract

Our aim in this paper is to introduce the results on the Betti numbers of local cohomology modules, which are obtained from the recent results on a category of cofinite modules (cf. [12], [14]) as a natural consequence. In this paper, we shall summarize those as corollaries in terms of Betti numbers.

Key Words：local cohomology, cofinite module, Betti number, Bass number.

We assume that all rings are commutative and noetherian with identity throughout this paper. Further we denote by $V(I)$ the set of all the prime ideals of A containing I, which is a closed subset with respect to the Zariski topology on the spectrum Spec A of A.

1. Introduction

In this section, we introduce classical results on our research. The following theorem is fundamental, due to Matlis and Grothendieck (cf. [15] and [6]).

Theorem 1 *Let A be a complete local ring, with maximal ideal $\mathfrak{m}$, and residue field $k=A/\mathfrak{m}$. Let $E=E_A(k)$ be an injective hull of k over A. For an A-module N, the following conditions are equivalent.*

(i) *N satisfies the descending chain condition (dcc);*
(ii) *N is a submodule of E^n, the direct sum of n copies of E, for some n;*
(iii) *There is an A-module M of finite type such that N is isomorphic to* $\mathrm{Hom}_A(M, E)$;
(iv) $\mathrm{Supp}_A N \subseteq V(\mathfrak{m})$ *and* $\mathrm{Hom}_A(k, N)$ *is of finite type;*
(v) $\mathrm{Supp}_A N \subseteq V(\mathfrak{m})$ *and* $\mathrm{Ext}^i_A(k, N)$ *is of finite type for all i;*
(vi) $\mathrm{Supp}_A N \subseteq V(\mathfrak{m})$ *and* $\mathrm{Hom}_A(N, E)$ *is of finite type.*

Proof. See [8] for the proof (See [9] also). □

In [8], the module N satisfying the above equivalent conditions is called *cofinite* (or $\mathfrak{m}$-*cofinite*).

It is natural to give a question whether Theorem 1 holds for a non-maximal ideal of A. The four questions were proposed in the paper [8, §2]. Especially the following was given:

Question 1 (Second Question) *Do the R-modules N satisfying the condition*

$(*)$ $\mathrm{Supp}_R(N) \subseteq V(J)$ and
$\mathrm{Ext}^j_R(R/J, N)$ is of finite type, for all j

form an Abelian subcategory of the category $\mathcal{M}(R)$ of all R-modules ? Here R is a regular ring and J is an ideal of R.

We denote by $\mathcal{M}(R, J)_{cof}$ the category of all R-modules N satisfying the condition $(*)$ according to [8]. Further an object N of $\mathcal{M}(R, J)_{cof}$ is called *J-cofinite* in this paper.

In [8, §3 An Example], Question 1 is answered negatively for an ideal generated by two elements and for that of dimension two. The example is as follows: Let R be the formal power series ring $k[x, y][[u, v]]$ over a polynomial ring $k[x, y]$, where k is a field. Let J be the ideal (u, v) of R, and M the R-module $R/(xv+yu)$. Then it was proved that the local cohomology module $\mathrm{H}^2_J(M)$ is not J-cofinite in [8, §3 An Example]. Even the dimension of the socle of that is not finite. Consequently, $\mathcal{M}(R, J)_{cof}$ is not Abelian for the ideal $J=(u, v)$, which is generated

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by two elements u, v and of dimension two.

Recently in [12] and [14], the above questions are answered affirmatively for an ideal of dimension one of a local ring and for a principal ideal of a ring. In this paper, we shall introduce the results as corollaries, obtained from the contributions in [12] and [14]. We shall concentrate the statements on those for Betti numbers, since those for Bass numbers have already appeared in several papers.

The following result is basic in this paper, which is due to Melkersson (cf. [17, Theorem 1.9, p. 420]):

Theorem 2 *Let A be a ring, and M an A-module. If M is I-cofinite for some ideal I of A, then all the Bass numbers $\mu_j(\mathfrak{p}, M)$ and all the Betti numbers $\beta_j(\mathfrak{p}, M)$ of M are finite for all $j \geq 0$. Namely, for each prime ideal $\mathfrak{p}$ of A and each integer $j \geq 0$, all the Bass numbers $\mu_j(\mathfrak{p}, M) = \mathrm{Ext}^j_{A_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})$ and all the Betti numbers $\beta_j(\mathfrak{p}, M) = \mathrm{Tor}_j^{A_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})$ are finite dimensional vector spaces over the residue field $k(\mathfrak{p})$ of the local ring $A_\mathfrak{p}$.*

2. The cases for ideals of codimension one over rings

The following was pointed out as a theorem in [14].

Theorem 3 *Let A be a ring, and I an ideal of A. If I is an ideal generated by one element of A up to radicals, then the subcategory $\mathcal{M}(A, I)_{cof}$ of $\mathcal{M}(A)$ is Abelian.*

Proof. See [14] for the proof. □

The following theorem is found in [11], concerning the question of Grothendieck [7] and the first question of Hartshorne [8].

Theorem 4 *Let A be a ring, and I an ideal of A. Let M be an A-module of finite type. If I is an ideal generated by one element of A up to radicals, then the local cohomology module $H^j_I(M)$ is I-cofinite for all $j \geq 0$.*

Proof. See [11] for the proof. □

Combining Theorem 3 with Theorem 4, we can get the following corollary.

Corollary 5 *Let A be a ring, and I an ideal of A. Let M, N be A-modules of finite type. Suppose that the projective dimension of M is finite. If I is an ideal generated by one element of A up to radicals, then the generalized local cohomology module $H^j_I(M, N)$ is I-cofinite for all $j \geq 0$. Further all the Betti numbers (and all the Bass numbers) of the generalized local cohomology module $H^j_I(M, N)$ are finite for all $j \geq 0$.*

Proof. First we note that there is a spectral sequence:

$$E_2^{p,q} = \mathrm{Ext}^p_R(M, H^q_I(N)) \underset{p}{\Longrightarrow} H^{p+q} = H^{p+q}_I(M, N).$$

Since the projective dimension of M is finite, one can see that $E_2^{p,q} = \mathrm{Ext}^p_R(M, H^q_I(N))$ are I-cofinite for all $p \geq 0$ and $q \geq 0$ by induction on $\mathrm{pd}_A(M)$. Here the first step of the induction follows from Theorem 4, which is the case for the projective dimension of M is zero. We also remark that a projective module is a direct summand of a free module.

Now let us show that $H^t = H^t_I(M, N)$ are I-cofinite for all $t \geq 0$. From Theorem 3, it follows that all $E_r^{p,q}$ are I-cofinite for all $p \geq 0$, $q \geq 0$ and $r \geq 2$, and all the kernel and image of the differentials of E_r-terms are I-cofinite for all $r \geq 2$. So it holds that all the limit terms $E_\infty^{p,q}$ and all A-modules appearing in the filtration of each abutment term H^{p+q} are I-cofinite for all $p \geq 0$ and $q \geq 0$. Therefore all the abutment terms H^t are I-cofinite for all $t \geq 0$, as required (cf. [1, Theorem 2.9]).

For the second statement, it follows from Theorem 2. □

3. The cases for ideals of dimension one over local rings

Recently the following was proved in [12].

Theorem 6 *Let A be a noetherian local ring, and I an ideal of A. If I is an ideal of dimension one, then the subcategory $\mathcal{M}(A, I)_{cof}$ of $\mathcal{M}(A)$ is Abelian.*

Proof. See [12] for the proof. □

The following theorem is the result of works by several authors (cf. [3], [9], [4], [5], and [19]) concerning the question of Grothendieck [7] and the first question of Hartshorne [8].

Theorem 7 *Let A be a ring, and I an ideal of A. Let M be an A-module of finite type. If I is an ideal of A of* *dimension one, then the local cohomology module* $H_I^j(M)$ *is I-cofinite for all* $j \geq 0$.

Combining Theorem 7 with Theorem 6, we obtain the following corollaries.

Corollary 8 *Let R be a regular ring, and J an ideal of R. Let M be an R-module of finite type. If J is of dimension one, then all the Betti numbers of* $D_J^j(M)$ *are finite for all* $j \geq 0$, *where* $D_J^j(M) = H^j(D_J(M))$ *is the cohomology module of the complex* $D_J(M)$ *applying the J-dualizing functor* $D_J(-)$ *to M (See [16] for the definition of J-dualizing functors).*

Proof. We may assume that R is a regular local ring of dimension d, after localizing R by a prime ideal of R. Now $D_J^j(M)$ is I-cofinite for all $j \geq 0$ by [12, Corollary 2]. So the assertion follows from Theorem 2. □

Corollary 9 *Let A be a ring, and I an ideal of A. Let M, N be A-modules of finite type with* $\mathrm{pd}_A(M) < \infty$. *If I is an ideal of A of dimension one, then all the Betti numbers of the generalized local cohomology module* $H_I^j(M, N)$ *are finite for all* $j \geq 0$.

Proof. We may assume that A is a local ring, after localizing A by a prime ideal of A. Now $H_I^j(M, N)$ is I-cofinite for all $j \geq 0$, repeating the same argument as the proof of Corollary 5. So the assertion follows from Theorem 2. □

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